arXiv:1304.7117v1 [math.RA] 26 Apr 2013

HOMOLOGICAL SMOOTHNESS AND DEFORMATIONS OF GENERALIZED WEYL ALGEBRAS

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Abstract. It is proved that a generalized Weyl algebra $A = \mathbf{k}[z](\sigma, \varphi(z))$ is homologically smooth if and only if the polynomial $\varphi(z)$ has no multiple roots, and thus the Van den Bergh duality holds. Moreover, their infinitesimal deformations are studied when $\mathbf{k}$ is of characteristic zero and A is homologically smooth. Under the technical processing of $\mathbf{k}$, A is a deformation of a certain commutative algebra $\mathfrak{A}$. And except a very few cases, there is a deformation of A back to $\mathfrak{A}$.

1. Introduction

During the development of algebra, an impetus is to introduce and study noncommutative objects with commutative background. Among these noncommutative objects, a class of algebras—generalized Weyl algebras, introduced by Bavula in [1]—have been studied from different points of view. There are many examples related to rings of differential operators or quantum groups, such as the usual Weyl algebras, quantum planes, and quantum spheres.

Roughly speaking, a generalized Weyl algebra A over a field $\mathbf{k}$ is determined by a $\mathbf{k}$-algebra B, as well as an algebra automorphism σ of B and a central element $a \in B$, denoted by $A = B(\sigma, a)$. Starting with the same B, people can construct generalized Weyl algebras A having different ring-theoretic and/or homological properties. These properties depend on the order of σ, the principal ideal (a), and the characteristic of $\mathbf{k}$. In particular, when B is the polynomial algebra $\mathbf{k}[z]$ and $a = \varphi(z)$, it was illustrated in [1] that the global dimension of A is probably equal to 1, 2 or ∞, and the latter occurs if and only if $\varphi(z)$ admits a multiple root (see also [2], [9]). On the other hand, the Hochschild (co)homology of A was calculated by [4] and [15]. Coincidentally, it is also equivalent to $\sup\{\, n \mid H^n(A, A) \neq 0 \,\} = \infty$ that $\varphi(z)$ admits a multiple root. As a consequence, it is pointed out in [4] that to assure duality between Hochschild homology and cohomology, the A^e-module A should have finite projective dimension (also called Hochschild cohomological dimension, denoted by

2010 *Mathematics Subject Classification.* 16E10, 16E40, 16S80.

Key words and phrases. generalized Weyl algebra, homologically smooth, deformation.

The author acknowledges the support of the European Union for ERC grant No 257004-HHNcdMir.

Hcdim). The fact actually indicates that an algebra is of necessity homologically smooth if the Van den Bergh duality holds.

An algebra A is said to be homologically smooth if A has a finitely generated projective resolution of finite length as an A^e-module. Van den Bergh proved in [17] that over a homologically smooth algebra A, if there is an invertible bimodule U and $d \geq 0$ such that $\mathrm{RHom}_{A^e}(A, A^e) \cong U^{-1}[-d]$ in the derived category $\mathbf{D}^b(A^e)$, then the duality $H^\bullet(A, M) \cong H_{d-\bullet}(A, U \otimes_A M)$ holds for any A-bimodule M. Van den Bergh duality holds for plenty of algebras, such as noetherian Artin-Schelter regular connected graded algebras, noetherian Artin-Schelter regular Hopf algebras, and some filtered algebras. Although [4] shows the condition $\varphi(z)$ has no multiple roots is necessary for the duality between Hochschild homology and cohomology, it does not tell us if the condition also guarantees the Van den Bergh duality for any bimodule M.

The problem boils down to the homological smoothness of generalized Weyl algebra $A = \mathbf{k}[z](\sigma, \varphi(z))$. Some simple examples of generalized Weyl algebras, say Weyl algebra $A_1(\mathbf{k})$, quantum 2-plane $\mathbf{k}_q[x, y]$ and the localization $\mathbf{k}_q[x, y^{\pm 1}]$, are all homologically smooth. In [10], Krähmer proved the standard quantum 2-sphere is Artin-Schelter Gorenstein and homologically smooth. He also asked whether the non-standard ones are also homologically smooth. The above results are obtained according to the individual structures of these algebras. In this paper, we make use of their common features to give a sufficient and necessary condition for a class of generalized Weyl algebras being homologically smooth. Our tool is called homotopy double complex, which seems feasible to have other applications. Under the condition, the generalized Weyl algebras are twisted Calabi-Yau (Lemma 4.2 and Theorem 4.3).

Theorem 1.1. *Let $A = \mathbf{k}[z](\sigma, \varphi(z))$ be a generalized Weyl algebra with $\sigma(z) = \lambda z + \eta$. The following are equivalent:*

(1) *The Hochschild cohomological dimension of A is 2;*
(2) *The global dimension of A is finite;*
(3) *$\varphi(z)$ has no multiple roots.*

When these conditions are satisfied, A is ν-twisted Calabi-Yau of dimension 2 where $\nu(x) = \lambda x$, $\nu(y) = \lambda^{-1} y$ and $\nu(z) = z$.

In particular, quantum 2-spheres are all homologically smooth. This gives a positive answer to [10, Question 2].

This paper is also dedicated to the deformation of homologically smooth generalized Weyl algebras. Deformation theory is a system studying how an object in a certain category of spaces can be varied in dependence of the points of a parameter space. It deals with the structure of families of objects like varieties, singularities,

vector bundles, presheaves, algebras or differentiable maps. Deformation problems appear in various areas of mathematics, in particular in algebra, algebraic and analytic geometry, and mathematical physics.

We care about deformation of associative algebras. Historically, the theme has its root in the work of Gerstenhaber [5]. There is an intimate connection between deformation theory and Hochschild cohomology. In particular, the second Hochschild cohomology group $H^2(A,A)$ may be interpreted as the group of the first order deformations of A. A class of generalized Weyl algebras were studied as deformations of type-A Kleinian singularties [9]. In this paper, we give an explanation how to treat a noncommutative generalized Weyl algebra $A = \mathbf{k}[z](\sigma, \varphi(z))$ as a deformation of a commutative algebra $\mathfrak{A}$ (A is a quotient of the free algebra $\mathbf{k}\langle x, y, z\rangle$ and so $\mathfrak{A}$ is a quotient of the polynomial algebra $\mathbf{k}_0[x, y, z]$ over a subfield $\mathbf{k}_0 \subset \mathbf{k}$), and then compute the second cohomology group of A which can be viewed as parameterizing "deformation of deformation".

The Hochschild cohomology of $\mathbf{k}[z](\sigma, \varphi(z))$ has been computed in [4] and [15] already, without the homological smoothness condition. Their computation is due to spectral sequence argument, and the dimensions of Hochschild cohomology groups in each degree are given. However, we are not only interested in the cohomology groups itself, but the Hochschild 2-cocycles. It seems somewhat hopeless to find the cocycles from their results. Moreover, they leave out several cases. Some important algebras, such as quantum 2-planes and the standard quantum 2-spheres, are not covered.

We hope to solve it in a different way. In fact, our goal is the deformation of the homologically smooth algebras. By virtue of Van den Bergh duality, we compute the second Hochschild cohomology group in a unified method when $\mathbf{k}$ is of characteristic zero, including the exceptional cases in [4] and [15]. By comparison of the nice projective resolution constructed in Sect. 3 and the bar resolution, the Hochschild 2-cocycles are found (Theorem 5.4).

Theorem 1.2. *Let $A = \mathbf{k}[z](\sigma, \varphi(z))$ be a homologically smooth generalized Weyl algebra, and let K be any element in the vector space* (5.7) *or* (5.9)*. The equivalence class of first order deformations of A corresponds to the 2-cocycle $F_K\colon A \times A \to A$ given by*

$$\begin{aligned}
F_K(z^p x^q, z^i x^j) &= -\lambda z^p(\Delta^\nu(x^q)\cdot K)xD(z^i)x^j,\\
F_K(z^p x^q, z^i y^j) &= -\lambda z^p(\Delta^\nu(x^q)\cdot K)xD(z^i)y^j\\
&\qquad\qquad - \sum_{k=1}^{\gamma+1} z^p(\Delta^\nu(x^{q-k+1})\cdot K)D(\varphi\circ\sigma)(xy)^{-1}(x^k z^i y^j),\\
F_K(z^p y^q, z^i x^j) &= z^p y(\Delta^\nu(y^q)\cdot K)D(z^i)x^j
\end{aligned}$$

$$+\sum_{k=1}^{\gamma} z^p y(\Delta^\nu(y^{q-k})\cdot K)D(\varphi)(yx)^{-1}(y^k z^i x^j),$$
$$F_K(z^p y^q, z^i y^j) = z^p y(\Delta^\nu(y^q)\cdot K)D(z^i)y^j,$$

where $D = \mathrm{d}/\mathrm{d}z$, $\gamma = \min\{q-1, j\}$, Δ^ν *is an operator (see subsection 5.4).*

Another advantage of such algebras is the vanishing of obstructions, that is, the third Hochschild cohomology group is zero. This is a nice condition in deformation theory, which implies every 2-cocycle is integrable. In particular, in the quantum case, we find an interesting phenomenon—return trip. There is an integration such that the formal deformation of A is $\mathfrak{A}$ up to extension of ground field, or, such a specific "deformation of deformation" of $\mathfrak{A}$ is $\mathfrak{A}$ itself. But in the classical case, return trips appear if and only if $\deg\varphi(z) \geq 2$. This implies the Weyl algebra $A_1(\mathbf{k})$ cannot return to $\mathbf{k}[x, y]$.

This paper is organized as follows. In Sect. 2, besides reviewing the definitions of generalized Weyl algebra and formal deformation, we introduce the homotopy double complexes as well as the associated total complex. In Sect. 3, we construct a homotopy double complex for a class of generalized Weyl algebras and prove that the associated total complex is a bimodule projective resolution. In Sect. 4, using the projective resolution, we find a sufficient and necessary condition that a generalized Weyl algebra is homologically smooth. In Sect. 5, deformations of homologically smooth generalized Weyl algebras are studied. We illustrate how to deform a commutative algebra to be a generalized Weyl algebra, establish the Van den Bergh duality, and list all equivalence classes of the first order infinitesimal deformations. Finally, we find special formal deformations which offer a way back to commutative algebras.

2. Preliminaries

Throughout, $\mathbf{k}$ is a field and all vector spaces and algebras are over $\mathbf{k}$ unless stated otherwise. Unadorned $\otimes$ means $\otimes_{\mathbf{k}}$. Let A be an algebra and M an A-bimodule. The group of all algebra automorphisms of A is denoted by $\mathrm{Aut}(A)$. For any f, $g \in \mathrm{Aut}(A)$, denote by ${}^f M^g$ the A-bimodule with the same ground vector space as M and the A-action twisted by f and g, that is, $a_1 \cdot m \cdot a_2 = f(a_1)mg(a_2)$ for any a_1, $a_2 \in A$, $m \in M$. If one of f and g is the identity map, it is usually omitted.

Let A^{op} be the opposite algebra of A and $A^e = A \otimes A^{\mathrm{op}}$ the enveloping algebra of A. The A-bimodule M can be viewed as a left A^e-module in a natural way. We use the following notations: $(a_1 \otimes a_2)\cdot m = a_1 m a_2$ for any a_1, $a_2 \in A$ and $m \in M$.

2.1. **Generalized Weyl algebras.** In this subsection, we recall the definition of generalized Weyl algebras given by Bavula in [1].

Definition 2.1. Suppose B is an algebra. For a central element $a \in \mathcal{Z}(B)$ and an algebra automorphism $\sigma \in \mathrm{Aut}(B)$, the *generalized Weyl algebra* (GWA for short) $A = B(\sigma, a)$ is by definition generated by 2 variables x and y over B subject to

$$xb = \sigma(b)x,\ yb = \sigma^{-1}(b)y,\ \forall\, b \in B,$$
$$yx = a,\ xy = \sigma(a).$$

Denote

$$x_i = \begin{cases} x^i, & \text{if } i \geq 0, \\ y^{-i}, & \text{if } i < 0, \end{cases}$$

then $A = \bigoplus_{i \in \mathbb{Z}} Bx_i$, and $Bx_i = x_iB$.

There are various algebras belong to the class of GWA, such as the usual Weyl algebra $A_1(\mathbf{k})$, the enveloping algebra $U(\mathfrak{sl}(2,\mathbf{k}))$ as well as its primitive factors $U(\mathfrak{sl}(2,\mathbf{k}))/(C - \lambda)$ where C is the Casimir element and $\lambda \in \mathbf{k}$, the quantum 2-spheres, and so on (see [2]).

Some properties of GWA have been studied. But the literature on their homological smoothness is quite limited. Recall that an algebra A is said to be homologically smooth if A admits a finitely generated projective resolution of finite length as a left A^e-module. One aim of this paper is to construct projective resolutions for a class of GWA A and to give a sufficient and necessary condition under which A is homologically smooth. In fact, it turns out that A is twisted Calabi-Yau.

Definition 2.2. Suppose A is an algebra and $\nu \in \mathrm{Aut}(A)$. Then A is called ν-*twisted Calabi-Yau* of dimension d for some $d \in \mathbb{N}$ if A is homologically smooth, and

$$\mathrm{Ext}^i_{A^e}(A, A^e) \cong \begin{cases} 0, & \text{if } i \neq d, \\ A^\nu, & \text{if } i = d \end{cases}$$

as A^e-modules, where the left A^e-module structure of A^e is used to compute the homology and the right one is retained, inducing the A^e-module structures on the homology groups. In this case, ν is called the *Nakayama automorphism* of A.

Next, we give some algebras playing a subsidiary role.

Let $B_1 = B[x;\sigma]$ be the skew-polynomial extension. Then we extend σ^{-1} to the automorphism σ_1 of B_1 by sending x to x, and define the σ_1-derivation $\delta_1 \colon B_1 \to B_1$ by

$$\delta_1|_B = 0,\ \delta_1(x) = a - \sigma(a).$$

Similarly, $B_2 = B[y;\sigma^{-1}]$ is a skew-polynomial extension too. Extend σ to the automorphism σ_2 of B_2 by sending y to y, and define the σ_2-derivation $\delta_2 \colon B_2 \to B_2$ by

$$\delta_2|_B = 0,\ \delta_2(y) = \sigma(a) - a.$$

Let $B_3 = B_1[y; \sigma_1, \delta_1]$. It is easy to check $B_3 = B_2[x; \sigma_2, \delta_2]$ and that $\omega := xy - \sigma(a)$ is a central regular element in B_3, and $A \cong B_3/\omega B_3$.

By [11], the three algebras are all twisted Calabi-Yau provided that so is B.

2.2. **Spectral sequence of a homotopy double complex.** Let us introduce the notions of homotopy double complexes and the associated total complexes.

Suppose that $\mathcal{A}$ is an abelian category. Consider a family of objects $\{C^{pq}\}_{p,q\in\mathbb{Z}}$ in $\mathcal{A}$ together with morphisms d_v, d_h, s of degrees $(0,1)$, $(1,0)$, $(2,-1)$ respectively.

Definition 2.3. A 4-tuple $(C^{\cdot\cdot}, d_v, d_h, s)$ is called a *homotopy double cochain complex* if

$$d_v^2 = 0,\ d_h d_v + d_v d_h = 0,\ d_h^2 + d_v s + s d_v = 0,\ d_h s + s d_h = 0,\ s^2 = 0. \tag{2.1}$$

Associated to a homotopy double cochain complex, the *total complex* $(\operatorname{Tot} C^{\cdot\cdot}, d)$ is defined by $(\operatorname{Tot} C^{\cdot\cdot})^n = \bigoplus_{p+q=n} C^{pq}$ and $d = d_v + d_h + s$.

Homotopy double chain complexes and the associated total complexes can be defined similarly.

It is easy to see that by letting $s = 0$, a homology double complex as well as the associated total complex is exactly the usual double complex as well as the usual total complex.

The filtration by columns

$$(F^n C^{\cdot\cdot})^{pq} = \begin{cases} C^{pq}, & \text{if } p \geq n, \\ 0, & \text{if } p < n \end{cases}$$

makes $\operatorname{Tot} C^{\cdot\cdot}$ be a filtered complex. This gives rise to a spectral sequence E_r^{pq}, starting with $E_0^{pq} = C^{pq}$. The differentials d_0 are just d_v of $C^{\cdot\cdot}$, so $E_1^{pq} = H_v^q(C^{p,\cdot})$. The differentials d_1 are induced by d_h of $C^{\cdot\cdot}$ since d_h^2 is null homotopy, so we have $E_2^{pq} = H_h^p H_v^q(C^{\cdot\cdot})$.

If $C^{\cdot\cdot}$ vanishes in the fourth quadrant, then the filtration is bounded. Thus $E_2^{pq} = H_h^p H_v^q(C^{\cdot\cdot}) \Rightarrow H^{p+q}(\operatorname{Tot} C^{\cdot\cdot})$.

Similarly, if we start with a homotopy double chain complex $C_{\cdot\cdot}$ vanishing in the second quadrant, then we have a convergent spectral sequence $E_{pq}^2 = H_p^h H_q^v(C_{\cdot\cdot}) \Rightarrow H_{p+q}(\operatorname{Tot} C_{\cdot\cdot})$.

Remark 2.4. Unlike with the usual double complexes, one fails to endow the total complex of a homotopy double complex with the filtration by rows because the differentials would not be well defined.

Suppose that there exist enough projective objects in $\mathcal{A}$. Let $C_\cdot$ be a chain complex with $C_p = 0$ for all $p < 0$, and $(\mathcal{P}_{\cdot\cdot}, d^v, d^h, s)$ be a homotopy double complex in the first quadrant. If for each p, $\mathcal{P}_{p,\cdot}$ is a projective resolution of C_p, then it follows from the spectral sequence that $\operatorname{Tot} \mathcal{P}_{\cdot\cdot}$ is a projective resolution of $C_{\cdot\cdot}$

2.3. **Deformation of an associative algebra.** This subsection is devoted to a review of formal deformations. The reader is referred to the survey [6] for details.

Let A be an algebra, and $(C^{\cdot}(A,A), \mathsf{b})$ the Hochschild cochain complex of A. Denote by $\mathbf{k}[[t]]$ the ring of formal power series in an indeterminate t, and by $A[[t]]$ the $\mathbf{k}[[t]]$-module of formal power series $\sum_{n=0}^{\infty} a_n t^n$ with coefficients in A, or equivalently $A[[t]] = A \widehat{\otimes} \mathbf{k}[[t]]$. Given a family of $\mathbf{k}$-bilinear maps $F_n \colon A \times A \to A$, $n \geq 1$, one obtains a $\mathbf{k}$-bilinear map $* \colon A \times A \to A[[t]]$ defined by

$$u * v = uv + F_1(u,v)t + F_2(u,v)t^2 + \cdots .$$

A *formal deformation* of A is such a $*$ that the extended $\mathbf{k}[[t]]$-bilinear map $A[[t]] \times A[[t]] \to A[[t]]$ determines an associative multiplication on $A[[t]]$. In this case, the maps F_n satisfy

$$\sum_{i=1}^{n-1} F_i \bullet F_{n-i} = \mathsf{b} F_n \tag{2.2}$$

where $F_i \bullet F_{n-i} \in C^3(A,A)$ is defined by

$$F_i \bullet F_{n-i}(a_1,a_2,a_3) = F_i(F_{n-i}(a_1,a_2),a_3) - F_i(a_1, F_{n-i}(a_2,a_3)).$$

Two formal deformations $*$ and $*'$ are said to be *equivalent* if there is a $\mathbf{k}[[t]]$-algebra isomorphism $G \colon (A[[t]], *) \to (A[[t]], *')$ such that

$$G(u) \equiv u \mod tA[[t]]$$

for all $u \in A$.

Replace $\mathbf{k}[[t]]$, $A[[t]]$ by the truncated polynomial ring $\mathbf{k}[t]/(t^{n+1})$, $A \otimes \mathbf{k}[t]/(t^{n+1})$ respectively for all $n \geq 1$, the *n-th order deformation* and equivalence relation can be defined similarly. These deformations are collectively called *infinitesimal deformations.*

If we view each F_n as an element in the Hochschild cochain module $C^2(A,A)$, then F_1 should be a 2-cocycle by the associative law. Moreover, if F_1' is another 2-cocycle, then F_1, F_1' represent the same cohomology class in $H^2(A,A)$ if and only if F_1' appears in a formal deformation $*'$ equivalent to $*$. A natural question is: is any 2-cocycle lifted to a formal deformation? The answer is no in general. In fact, there is a bijection between $H^2(A,A)$ and the family of equivalence classes of first order deformations.

Starting with $F_1 \in Z^2(A,A)$, one may show $\operatorname{Sq} F_1 := F_1 \bullet F_1 \in Z^3(A,A)$. Let F_0 be the multiplication on A, and so $F_0 + F_1 t$ defines an associative multiplication on $A \otimes \mathbf{k}[t]/(t^2)$. By (2.2), $\operatorname{Sq} F_1 = \mathsf{b} F_2$ for some F_2 if and only if $F_0 + F_1 t + F_2 t^2$ defines an associative multiplication on $A \otimes \mathbf{k}[t]/(t^3)$. The cohomology class $[\operatorname{Sq} F_1] \in H^3(A,A)$ is vividly called the *primary obstruction* to integrating F_1. Generally, if there is an $(n-1)$-st order deformation $F_0 + F_1 t + \cdots + F_{n-1} t^{n-1}$, then the left-hand side of

(2.2) is always a 3-cocycle and it is a coboundary if and only if there is an n-th order deformation $F_0 + F_1 t + \cdots + F_{n-1}t^{n-1} + F_n t^n$. If all obstructions can be passed successfully, i.e., $[F_1 \bullet F_n + F_2 \bullet F_{n-1} + \cdots + F_n \bullet F_1] = 0$ in $H^3(A, A)$ for all $n \geq 1$, such an F_1 is said to be *integrable*.

It is a difficult problem to decide when a 2-cocycle is integrable, unless luckily, one has $H^3(A, A) = 0$. In Sect. 4, we will show the homologically smooth GWA is such an algebra.

2.4. **Some results in linear algebra.** Let λ, $z_1, \ldots, z_l$ be some numbers in $\mathbf{k}$ with $\lambda \neq 0$. Let e be the order of λ if λ is a root of unity, or zero if not. Namely, $e\mathbb{N} = \{n \in \mathbb{N} \mid \lambda^n = 1\}$. Denote by $V(z_1, \ldots, z_l)$ the following Vandermonde Matrix indexed by $\mathbb{N}^+$

$$\begin{pmatrix} 1 & z_1 & z_1^2 & z_1^3 & \cdots \\ 1 & z_2 & z_2^2 & z_2^3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \\ 1 & z_l & z_l^2 & z_l^3 & \cdots \end{pmatrix}.$$

Let $V_\lambda(z_1, \ldots, z_l)$ be the sub-matrix obtained by picking the $(ke+2)$-nd column of $V(z_1, \ldots, z_l)$ as the $(k+1)$-st column for all $k \in \mathbb{N}$. Then we have

$$V_\lambda(z_1, \ldots, z_l) = \operatorname{diag}(z_1, \ldots, z_l) V(z_1^e, \ldots, z_l^e).$$

Let $\varphi(z) \in \mathbf{k}[z]$ be a polynomial with roots $z_1, \ldots, z_l$. Denote by $R(\varphi, \lambda)$ the rank of $V_\lambda(z_1, \ldots, z_l)$. Identify any constant $c \in \mathbf{k}$ with the maximal ideal $(z - c)$ in $\mathbf{k}[z]$. Then $c \mapsto \lambda c$ determines a cyclic group action on the maximum spectrum $\operatorname{MSpec} \mathbf{k}[z]$. Clearly, (z) is a fixed point, no matter whether λ is a root of unity. The ideal $(\varphi(z))$ is decomposed into the intersection of a finite number of primary ideals uniquely. Consider the orbits of the associated prime ideals (in fact they are maximal) that differ from (z) in $\operatorname{MSpec} \mathbf{k}[z]$. Then $R(\varphi, \lambda)$ is equal to the number of these orbits.

For convention, let $R(\varphi, \lambda) = 0$ if $\varphi(z)$ is a nonzero constant.

3. Projective resolutions

3.1. **A complex.** For any GWA $A = B(\sigma, a)$, we will first construct a complex of A^e-modules as the cornerstone of the paper.

Proposition 3.1. *Suppose that $a \in B$ is not a zero-divisor. Let $C.$ be the chain complex of A^e-modules with $C_i = 0$ for all $i < 0$, $C_0 = A \otimes_B A$, $C_i = (A^\sigma \otimes_B A) \oplus (A \otimes_B {}^\sigma A)$ for all positive odd i, and $C_i = (A \otimes_B A) \oplus (A^\sigma \otimes_B {}^\sigma A)$ for all positive even i, whose differentials $d_i \colon C_i \to C_{i-1}$ are defined by*

$$d_1(1 \otimes 1, 0) = x \otimes 1 - 1 \otimes x, \qquad d_1(0, 1 \otimes 1) = y \otimes 1 - 1 \otimes y,$$

for all $j > 0$,

$$d_{2j}(1 \otimes 1, 0) = (y \otimes 1, 1 \otimes x), \qquad d_{2j}(0, 1 \otimes 1) = (1 \otimes y, x \otimes 1),$$
$$d_{2j+1}(1 \otimes 1, 0) = (x \otimes 1, -1 \otimes x), \quad d_{2j+1}(0, 1 \otimes 1) = (-1 \otimes y, y \otimes 1).$$

Then $H_i(C) = 0$ *for all* $i \neq 0$ *and* $H_0(C) = A$.

Proof. It is routine to check these maps are well defined and are differentials. Clearly, $H_0(C) = A$. Now let us prove the exactness in degree one.

Since

$$A^\sigma \otimes_B A = \bigoplus_{i,\, j \in \mathbb{Z}} x_i B^\sigma \otimes_B B x_j \cong \bigoplus_{i,\, j \in \mathbb{Z}} x_i B^\sigma x_j,$$

any element in $A^\sigma \otimes_B A$ can be expressed to be $\sum_{i,\,j} x_i \otimes b_{ij} x_j$ uniquely with $b_{ij} \in B$. Similar for $A \otimes_B {}^\sigma\! A$, $A \otimes_B A$, and $A^\sigma \otimes_B {}^\sigma\! A$. If $(\sum_{i,\,j} x_i \otimes b_{ij} x_j, \sum_{p,\,q} x_p \otimes c_{pq} x_q) \in \operatorname{Ker} d_1$, then

$$0 = \sum_{i,\,j} x_i x \otimes b_{ij} x_j - \sum_{i,\,j} x_i \otimes x b_{ij} x_j + \sum_{p,\,q} x_p y \otimes c_{pq} x_q - \sum_{p,\,q} x_p \otimes y c_{pq} x_q.$$

Let (i', j') be the greatest index with respect to the lexicographic order on $\mathbb{Z} \times \mathbb{Z}$ such that $b_{i'j'} \neq 0$, and (p', q') be the greatest index such that $c_{p'q'} \neq 0$. Then we have $x_{i'} x \otimes b_{i'j'} x_{j'} = x_{p'} \otimes y c_{p'q'} x_{q'}$. It follows that $(p', q') = (i' + 1, j' + 1)$, and

$$x_{i'} x \otimes b_{i'j'} x_{j'} = x_{i'+1} \otimes \sigma^{-1}(c_{i'+1, j'+1}) y x_{j'+1}. \tag{3.1}$$

Using the case-by-case argument, we deduce from (3.1) that

$$(x_{i'} \otimes b_{i'j'} x_{j'}, x_{p'} \otimes c_{p'q'} x_{q'}) = \begin{cases} x_{i'}(1 \otimes y, x \otimes 1) c_{p'q'} x_{q'}, & \text{if } i' \geq 0, \\ x_{p'}(y \otimes 1, 1 \otimes x) b_{i'j'} x_{j'}, & \text{if } i' \leq -1. \end{cases}$$

In either case, the highest term lies in $\operatorname{Im} d_2$. It follows that $\operatorname{Ker} d_1 = \operatorname{Im} d_2$.

For other positions, the exactness can be verified by the same method. □

3.2. **Construction of homotopy double complex.** As stated in Sect. 2, the projective resolution of a complex $(C_\cdot, \bar{d})$ can be constructed if there is a suitable homotopy double complex in the first quadrant. In particular, in Proposition 3.1, if such a $(\mathcal{P}_{\cdot\cdot}, d^v, d^h, r)$ exists, then $\operatorname{Tot} \mathcal{P}_{\cdot\cdot}$ is an A^e-projective resolution of A.

Using this homotopy double complex, the Hochschild cohomology of A can be computed. In fact, for any A-bimodule M, let $\mathcal{Q}^{pq} = \operatorname{Hom}_{A^e}(\mathcal{P}_{pq}, M)$, and ∂_v^{pq}, ∂_h^{pq}, s^{pq} be the maps obtained by letting act $\operatorname{Hom}_{A^e}(-, M)$ on $d^v_{p,q+1}$, $d^h_{p+1,q}$, $r_{p+2,q-1}$ respectively. Then $(\mathcal{Q}^{\cdot\cdot}, \partial_v, \partial_h, s)$ is also a homotopy double complex and

$$H^n(A, M) = H_n(\operatorname{Hom}_{A^e}(\operatorname{Tot} \mathcal{P}_{\cdot\cdot}, M)) = H^n(\operatorname{Tot} \mathcal{Q}^{\cdot\cdot}).$$

One question is: does such a homotopy double complex exist? Of course, we say yes definitely because the Cartan-Eilenberg resolution is a candidate. However, it is inconvenient to compute homology using Cartan-Eilenberg resolutions. In fact, we

only need to consider: if each C_p admits a nice projective resolution $(\mathcal{P}_{p,\cdot}, d^v)$, can we equip appropriate d^h, r such that $(\mathcal{P}_{\cdot\cdot}, d^v, d^h, r)$ is a homotopy double complex? Observe the five equations in (2.1). The first three are easy to satisfy, due to the Comparison Lemma. But it seems hard to make sure the last two hold.

A situation we are able to handle is when the projective dimensions of C_p's are at most 1. Let

$$0 \to \mathcal{P}_{p1} \xrightarrow{d_p^v} \mathcal{P}_{p0} \xrightarrow{\varepsilon_p} C_p \to 0$$

be a projective resolution. By the Comparison Lemma and the signs trick, each differential $\bar{d}_p \colon C_p \to C_{p-1}$ can be lifted to morphisms $d^h_{p0} \colon \mathcal{P}_{p0} \to \mathcal{P}_{p-1,0}$ and $d^h_{p1} \colon \mathcal{P}_{p1} \to \mathcal{P}_{p-1,1}$ such that $\varepsilon_{p-1} d^h_{p0} = \bar{d}_p \varepsilon_p$ and $d^v_{p-1} d^h_{p1} + d^h_{p0} d^v_p = 0$. Since $\varepsilon_{p-2} d^h_{p-1,0} d^h_{p0} = \bar{d}_{p-1} \bar{d}_p \varepsilon_p = 0$, we have $d^h_{p-1,0} d^h_{p0}(\mathcal{P}_{p0}) \subseteq \operatorname{Ker} \varepsilon_{p-2} = \operatorname{Im} d^v_{p-2}$. Since d^v_{p-2} is monic, there is a unique morphism $r_p \colon \mathcal{P}_{p0} \to \mathcal{P}_{p-2,1}$ such that $d^v_{p-2} r_p = -d^h_{p-1,0} d^h_{p0}$. It follows that

$$d^v_{p-3} d^h_{p-1,1} r_p = -d^h_{p-2,0} d^v_{p-2} r_p = d^h_{p-2,0} d^h_{p-1,0} d^h_{p0} = -d^v_{p-3} s_{p-2} d^h_{p0},$$

yielding $d^h_{p-1,1} r_p + r_{p-2} d^h_{p0} = 0$. Thus we obtain a homotopy double complex

$$\begin{tikzcd}
\mathcal{P}_{01} \arrow[d, "d_0^v"] & \mathcal{P}_{11} \arrow[l, "d^h_{11}"'] \arrow[d, "d_1^v"] & \mathcal{P}_{21} \arrow[l, "d^h_{21}"'] \arrow[d, "d_2^v"] & \mathcal{P}_{31} \arrow[l, "d^h_{31}"'] \arrow[d, "d_3^v"] & \mathcal{P}_{41} \arrow[l, "d^h_{41}"'] \arrow[d, "d_4^v"] & \cdots \arrow[l] \\
\mathcal{P}_{00} & \mathcal{P}_{10} \arrow[l, "d^h_{10}"] & \mathcal{P}_{20} \arrow[l, "d^h_{20}"] \arrow[ull, "r_2"] & \mathcal{P}_{30} \arrow[l, "d^h_{30}"] \arrow[ull, "r_3"] & \mathcal{P}_{40} \arrow[l, "d^h_{40}"] \arrow[ull, "r_4"] & \cdots \arrow[l]
\end{tikzcd}$$

From now on, let $B = \mathbf{k}[z]$. In this case the required homotopy double complex clearly exists. Suppose $a = \varphi(z) = \sum_{i=0}^{l} a_i z^i$ with $a_l \neq 0$, and $\sigma(z) = \lambda z + \eta$ with λ, $\eta \in \mathbf{k}$, $\lambda \neq 0$. Following [15], if $\sigma \neq \mathrm{id}$, there are two kinds of GWA essentially:

- classical case: $\lambda = 1$, $\eta \neq 0$,
- quantum case: $\lambda \neq 1$, $\eta = 0$.

Remark 3.2. In both cases, A^e-projective resolutions of A were constructed in [4] and [15] respectively. Their constructions are due to B_3. Here we obtain the same resolution in a different way, which works not only for $\mathbf{k}[z]$, but also for any formally smooth B, that is, $\operatorname{Hcdim} B \leq 1$. The homotopy double complex is then constructed in terms of noncommutative 1-form $\mathsf{d}b$ for all $b \in B$.

Choose B^e-projective resolutions of B to be

$$0 \to B \otimes B \xrightarrow{\delta_c} B \otimes B \to B \to 0$$

where $\delta_c(1 \otimes 1) = c(z \otimes 1 - 1 \otimes z)$ for any nonzero $c \in \mathbf{k}$. Then $\mathcal{P}_{00} = \mathcal{P}_{01} = A \otimes A$, $\mathcal{P}_{j0} = \mathcal{P}_{j1} = (A \otimes A)^{\oplus 2}$ for $j > 0$. The differential d_0^v is induced by δ_1 and others by $(\delta_1, \delta_{\lambda^{-1}})$.

Let us give the explicit expressions of d^h and r. It is clear that $d^h_{\cdot,0}$ can be defined by

$$\begin{aligned}
&d^h_{10}(1\otimes 1,0) = x\otimes 1 - 1\otimes x, && d^h_{10}(0,1\otimes 1) = y\otimes 1 - 1\otimes y,\\
&d^h_{2j,0}(1\otimes 1,0) = (y\otimes 1, 1\otimes x), && d^h_{2j,0}(0,1\otimes 1) = (1\otimes y, x\otimes 1),\\
&d^h_{2j+1,0}(1\otimes 1,0) = (x\otimes 1, -1\otimes x), && d^h_{2j+1,0}(0,1\otimes 1) = (-1\otimes y, y\otimes 1).
\end{aligned}$$

It follows that

$$\begin{aligned}
&d^h_{11}(1\otimes 1,0) = -x\otimes 1 + \lambda\otimes x, && d^h_{11}(0,1\otimes 1) = -y\otimes 1 + \lambda^{-1}\otimes y,\\
&d^h_{2j,1}(1\otimes 1,0) = (-y\otimes 1, -\lambda\otimes x), && d^h_{2j,1}(0,1\otimes 1) = (-\lambda^{-1}\otimes y, -x\otimes 1),\\
&d^h_{2j+1,1}(1\otimes 1,0) = (-x\otimes 1, \lambda\otimes x), && d^h_{2j+1,1}(0,1\otimes 1) = (\lambda^{-1}\otimes y, -y\otimes 1).
\end{aligned}$$

After that, let us construct $r_p\colon \mathcal{P}_{p0}\to\mathcal{P}_{p-2,1}$. Define the derivation $\Delta\colon \mathbf{k}[z]\to \mathbf{k}[z]\otimes\mathbf{k}[z]$ by $\Delta(z^k)=\sum_{i=1}^k z^{k-i}\otimes z^{i-1}$ for $k\geq 1$. For any $\mathbf{k}$-linear endomorphisms f, g of $\mathbf{k}[z]$, denote $(f\otimes g)\Delta$ by ${}^f\Delta^g$. By direct computation, we have

$$\begin{aligned}
d^h_{10}d^h_{20}(1\otimes 1,0) &= \varphi(z)\otimes 1 - 1\otimes\varphi(z),\\
d^h_{10}d^h_{20}(0,1\otimes 1) &= \sigma(\varphi(z))\otimes 1 - 1\otimes\sigma(\varphi(z)),\\
d^h_{2j,0}d^h_{2j+1,0}(1\otimes 1,0) &= (\sigma(\varphi(z))\otimes 1 - 1\otimes\varphi(z), 0),\\
d^h_{2j,0}d^h_{2j+1,0}(0,1\otimes 1) &= (0, \varphi(z)\otimes 1 - 1\otimes\sigma(\varphi(z))),\\
d^h_{2j+1,0}d^h_{2j+2,0}(1\otimes 1,0) &= (\varphi(z)\otimes 1 - 1\otimes\varphi(z), 0),\\
d^h_{2j+1,0}d^h_{2j+2,0}(0,1\otimes 1) &= (0, \sigma(\varphi(z))\otimes 1 - 1\otimes\sigma(\varphi(z))).
\end{aligned}$$

Thus r can be defined by

$$\begin{aligned}
&r_2(1\otimes 1,0) = -\Delta(\varphi), && r_2(0,1\otimes 1) = -\lambda^\sigma\Delta^\sigma(\varphi),\\
&r_{2j+1}(1\otimes 1,0) = (-{}^\sigma\Delta(\varphi),0), && r_{2j+1}(0,1\otimes 1) = (0,-\lambda\Delta^\sigma(\varphi)),\\
&r_{2j+2}(1\otimes 1,0) = (-\Delta(\varphi),0), && r_{2j+2}(0,1\otimes 1) = (0,-\lambda^\sigma\Delta^\sigma(\varphi)).
\end{aligned}$$

4. Homological smoothness

As a consequence of Sect. 3, for any A-bimodule M, we have the explicit form of $\mathcal{Q}^{\cdot\cdot}$ as follows, which is a diagram of B-bimodules except ∂_h^{00}, ∂_h^{01}, s^0.

$$\begin{tikzcd}
M \arrow[r, "\partial_h^{01}"] \arrow[drr, "s^0"] & {}^\sigma M\oplus M^\sigma \arrow[r, "\partial_h^{11}"] \arrow[drr, "s^1"] & M\oplus {}^\sigma M^\sigma \arrow[r, "\partial_h^{21}"] \arrow[drr, "s^2"] & {}^\sigma M\oplus M^\sigma \arrow[r, "\partial_h^{31}"] & M\oplus {}^\sigma M^\sigma \arrow[r] & \cdots \\
M \arrow[u, "\partial_v^0"] \arrow[r, "\partial_h^{00}"'] & {}^\sigma M\oplus M^\sigma \arrow[u, "\partial_v^1"] \arrow[r, "\partial_h^{10}"'] & M\oplus {}^\sigma M^\sigma \arrow[u, "\partial_v^2"] \arrow[r, "\partial_h^{20}"'] & {}^\sigma M\oplus M^\sigma \arrow[u, "\partial_v^3"] \arrow[r, "\partial_h^{30}"'] & M\oplus {}^\sigma M^\sigma \arrow[u, "\partial_v^4"] \arrow[r] & \cdots
\end{tikzcd}$$

with

$$\partial_h^{00}(m) = (xm-mx, ym-my),$$

$$\begin{aligned}
\partial_h^{2j-1,0}(m_1,m_2) &= (ym_1+m_2x, m_1y+xm_2),\\
\partial_h^{2j,0}(m_1,m_2) &= (xm_1-m_2x, -m_1y+ym_2),\\
\partial_h^{01}(m) &= (-xm+\lambda mx, -ym+\lambda^{-1}my),\\
\partial_h^{2j-1,1}(m_1,m_2) &= (-ym_1-\lambda m_2x, -\lambda^{-1}m_1y-xm_2),\\
\partial_h^{2j,1}(m_1,m_2) &= (-xm_1+\lambda m_2x, \lambda^{-1}m_1y-ym_2),\\
\partial_v^{0}(m) &= zm-mz,\\
\partial_v^{2j-1}(m_1,m_2) &= (\sigma(z)m_1-m_1z, \lambda^{-1}zm_2-\lambda^{-1}m_2\sigma(z)),\\
\partial_v^{2j}(m_1,m_2) &= (zm_1-m_1z, \lambda^{-1}\sigma(z)m_2-\lambda^{-1}m_2\sigma(z)),\\
s^0(m) &= (-\Delta(\varphi)\cdot m, -\lambda^\sigma\Delta^\sigma(\varphi)\cdot m),\\
s^j(m_1,m_2) &= (-\Delta(\varphi)\cdot m_1, -\lambda\Delta(\varphi)\cdot m_2).
\end{aligned}$$

Let $\partial^\cdot$ be the differentials of $\operatorname{Tot}\mathcal{Q}^{\cdot\cdot}$. Denote by $\varphi'(z)$ the formal derivative of $\varphi(z)$, and $\tilde{\varphi}(z)=\gcd(\varphi(z),\varphi'(z))$. Fix $\alpha(z)$, $\beta(z)\in\mathbf{k}[z]$ such that $\alpha(z)\varphi(z)+\beta(z)\varphi'(z)=\tilde{\varphi}(z)$.

Lemma 4.1. *One has* $\operatorname{Ker}\partial^3\cdot\tilde{\varphi}(z)\subseteq\operatorname{Im}\partial^2$. *In particular, if* $\varphi(z)$ *has no multiple roots, then* $H^3(\operatorname{Tot}\mathcal{Q}^{\cdot\cdot})=0$ *for all* A*-bimodules* M.

Proof. During the proof, we sometimes write a polynomial $h(z)$ as h, for simplicity.

Suppose that $(m_1,m_2,m_3,m_4)\in M\oplus{}^\sigma M^\sigma\oplus{}^\sigma M\oplus M^\sigma$ is an element in $\operatorname{Ker}\partial^3$. Then $\partial_h^{21}(m_1,m_2)+\partial_v^3(m_3,m_4)=0$ and $s^2(m_1,m_2)+\partial_h^{30}(m_3,m_4)=0$, that is,

$$-xm_1+\lambda m_2x+\sigma(z)m_3-m_3z=0, \tag{4.1}$$

$$m_1y-\lambda ym_2+zm_4-m_4\sigma(z)=0, \tag{4.2}$$

$$-\Delta(\varphi)\cdot m_1+ym_3+m_4x=0, \tag{4.3}$$

$$-\lambda\Delta(\varphi)\cdot m_2+m_3y+xm_4=0. \tag{4.4}$$

By induction, we obtain from (4.1), (4.2) that for any $j\geq 1$,

$$\begin{aligned}
\sigma(z)^jm_3-m_3z^j &= x(\Delta(z^j)\cdot m_1)-\lambda(\Delta(z^j)\cdot m_2)x,\\
m_4\sigma(z)^j-z^jm_4 &= (\Delta(z^j)\cdot m_1)y-\lambda y(\Delta(z^j)\cdot m_2).
\end{aligned}$$

Thus

$$\begin{aligned}
\Delta(\varphi)\cdot(m_3\beta) &= \sum_{i=1}^{l}a_i\sum_{j=1}^{i}\sigma(z)^{i-j}m_3\beta z^{j-1}\\
&= \sum_{i=1}^{l}\sum_{j=1}^{i}a_i\big(m_3z^{i-j}+x(\Delta(z^{i-j})\cdot m_1)-\lambda(\Delta(z^{i-j})\cdot m_2)x\big)\beta z^{j-1}\\
&= m_3\beta\varphi'+\sum_{i=2}^{l}\sum_{j=1}^{i-1}\sum_{k=1}^{i-j}a_i(xz^{i-j-k}m_1z^{j+k-2}-\lambda\sigma(z)^{i-j-k}m_2\sigma(z)^{j+k-2}x)\beta
\end{aligned}$$

$$
\begin{aligned}
&= m_3\beta\varphi' + \sum_{i=2}^{l}\sum_{j=1}^{i-1}\sum_{k=j+1}^{i} a_i(xz^{i-k}m_1z^{k-2} - \lambda\sigma(z)^{i-k}m_2\sigma(z)^{k-2}x)\beta \\
&= m_3\beta\varphi' + \sum_{i=2}^{l}\sum_{k=2}^{i}\sum_{j=1}^{k-1} a_i(xz^{i-k}m_1z^{k-2} - \lambda\sigma(z)^{i-k}m_2\sigma(z)^{k-2}x)\beta \\
&= m_3\beta\varphi' + \sum_{i=2}^{l}\sum_{k=2}^{i}(k-1)a_i(xz^{i-k}m_1z^{k-2} - \lambda\sigma(z)^{i-k}m_2\sigma(z)^{k-2}x)\beta \\
&= m_3\beta\varphi' + \sum_{i=1}^{l}\sum_{k=1}^{i} a_i(xz^{i-k}m_1D(z^{k-1}) - \lambda\sigma(z^{i-k})m_2\sigma(D(z^{k-1}))x)\beta \\
&= m_3\beta\varphi' + x(\Delta^D(\varphi)\cdot m_1)\beta - \lambda(\Delta^D(\varphi)\cdot m_2)\sigma(\beta)x,
\end{aligned}
$$

where $D = \mathrm{d}/\mathrm{d}z$. Similarly,

$$
\Delta(\varphi)\cdot(m_4\sigma(\beta)) = m_4\sigma(\beta\varphi') - (\Delta^D(\varphi)\cdot m_1)\beta y + \lambda y(\Delta^D(\varphi)\cdot m_2)\sigma(\beta). \tag{4.5}
$$

Let $n_1 = -m_3\beta \in {}^{\sigma}M$. Then the first component of $s^1(n_1, 0)$ is

$$
\begin{aligned}
&m_3\tilde{\varphi} - m_3\alpha yx + x(\Delta^D(\varphi)\cdot m_1)\beta - \lambda(\Delta^D(\varphi)\cdot m_2)\sigma(\beta)x \\
&= m_3\tilde{\varphi} + x(\Delta^D(\varphi)\cdot m_1)\beta - \big(m_3\alpha y + \lambda(\Delta^D(\varphi)\cdot m_2)\sigma(\beta)\big)x.
\end{aligned}
$$

Denote

$$
\begin{aligned}
n_3 &= -(\Delta^D(\varphi)\cdot m_1)\beta \in M, \\
n_4 &= -m_3\alpha y - \lambda(\Delta^D(\varphi)\cdot m_2)\sigma(\beta) \in {}^{\sigma}M^{\sigma}.
\end{aligned}
$$

Clearly, the first component of $s^1(n_1, 0) + \partial_h^{20}(n_3, n_4)$ equals $m_3\tilde{\varphi}$. The second one is equal to

$$
(\Delta^D(\varphi)\cdot m_1)\beta y - ym_3\alpha y - \lambda y(\Delta^D(\varphi)\cdot m_2)\sigma(\beta). \tag{4.6}
$$

Next, consider the difference between (m_1, m_2) and $\partial_h^{11}(n_1, 0) + \partial_v^2(n_3, n_4)$. Notice that

$$
z(\Delta^D(\varphi)\cdot m) - (\Delta^D(\varphi)\cdot m)z = \Delta(\varphi)\cdot m - m\varphi' \tag{4.7}
$$

for all $m \in M$. It follows that

$$
\begin{aligned}
zn_3 - n_3z &= m_1\beta\varphi' - (\Delta(\varphi)\cdot m_1)\beta, \\
\sigma(z)n_4 - n_4\sigma(z) &= m_3\alpha zy - \sigma(z)m_3\alpha y + \lambda m_2\sigma(\beta\varphi') - \lambda(\Delta(\varphi)\cdot m_2)\sigma(\beta).
\end{aligned}
$$

So by (4.1)–(4.4),

$$
\begin{aligned}
&\partial_h^{11}(n_1, 0) + \partial_v^2(n_3, n_4) \\
&= \big(ym_3\beta + m_1\beta\varphi' - (\Delta(\varphi)\cdot m_1)\beta,\ \lambda^{-1}m_3\beta y + \lambda^{-1}m_3\alpha zy \\
&\quad - \lambda^{-1}\sigma(z)m_3\alpha y + m_2\sigma(\beta\varphi') - (\Delta(\varphi)\cdot m_2)\sigma(\beta)\big) \\
&= \big(m_1\beta\varphi' - m_4x\beta,\ -\lambda^{-1}\sigma(z)m_3\alpha y + \lambda^{-1}m_3z\alpha y - \lambda^{-1}xm_4\sigma(\beta)
\end{aligned}
$$

$$
\begin{aligned}
&\quad + m_2\sigma(\tilde{\varphi}) - m_2x\alpha y\big)\\
&= \big(m_1\beta\varphi' - m_4x\beta,\ -\lambda^{-1}xm_1\alpha y - \lambda^{-1}xm_4\sigma(\beta) + m_2\sigma(\tilde{\varphi})\big)\\
&= \big(m_1\tilde{\varphi},\ m_2\sigma(\tilde{\varphi})\big) - \big(m_1\alpha yx + m_4\sigma(\beta)x,\ \lambda^{-1}xm_1\alpha y + \lambda^{-1}xm_4\sigma(\beta)\big)\\
&= (m_1, m_2)\cdot\tilde{\varphi} - \partial_h^{11}(0, -\lambda^{-1}m_1\alpha y - \lambda^{-1}m_4\sigma(\beta)).
\end{aligned}
$$

Denote $n_2 = -\lambda^{-1}m_1\alpha y - \lambda^{-1}m_4\sigma(\beta) \in M^\sigma$. Then

$$(m_1, m_2)\cdot\tilde{\varphi} = \partial_h^{11}(n_1, n_2) + \partial_v^2(n_3, n_4).$$

In order to finish the proof, we hope that the second component of $s^1(n_1, n_2)$ plus (4.6) is equal to $m_4\cdot\tilde{\varphi}$. In fact, since

$$
\begin{aligned}
\Delta(\varphi)\cdot(m_1\alpha y) &= (\Delta(\varphi)\cdot m_1)\alpha y\\
&= (ym_3 + m_4x)\alpha y = ym_3\alpha y + m_4xy\sigma(\alpha),
\end{aligned}
$$

together with (4.5), we have

$$
\begin{aligned}
&-\lambda\Delta(\varphi)\cdot n_2\\
&= ym_3\alpha y + m_4xy\sigma(\alpha) + m_4\sigma(\beta\varphi') - (\Delta^D(\varphi)\cdot m_1)\beta y\\
&\quad + \lambda y(\Delta^D(\varphi)\cdot m_2)\sigma(\beta)\\
&= m_4\sigma(\tilde{\varphi}) + ym_3\alpha y - (\Delta^D(\varphi)\cdot m_1)\beta y + \lambda y(\Delta^D(\varphi)\cdot m_2)\sigma(\beta)\\
&= m_4\cdot\tilde{\varphi} - (4.6).
\end{aligned}
$$

Therefore, $(m_1, m_2, m_3, m_4)\cdot\tilde{\varphi} = \partial^2(n_1, n_2, n_3, n_4)$. If $\varphi(z)$ has no multiple root, then $\tilde{\varphi}(z) = 1$. Thus $\operatorname{Ker}\partial^3 = \operatorname{Im}\partial^2$, and so $H^3(\operatorname{Tot}\mathcal{Q}^{\cdot\cdot}) = 0$. $\square$

Lemma 4.2. *There are isomorphisms of A^e-modules*

$$\operatorname{Ext}^i_{A^e}(A, A^e) \cong \begin{cases} 0, & \text{if } i \neq 2,\\ A^\nu, & \text{if } i = 2,\end{cases}$$

where the Nakayama automorphism ν of A is given by

$$\nu(x) = \lambda x,\ \nu(y) = \lambda^{-1}y,\ \nu(z) = z.$$

Or equivalently, the rigid dualizing complex over A is ${}^\nu A[2]$.

Proof. Notice that $B \subset B_1 \subset B_3$, $B \subset B_2 \subset B_3$ are chains of skew-polynomial extensions. By [11, Theorem 0.2] and [18, Proposition 1.1], the algebras B_1, B_2, B_3 are twisted Calabi-Yau and their Nakayama automorphisms are given by

$$
\begin{aligned}
&\nu_1(z) = \sigma^{-1}(z), && \nu_1(x) = \lambda x,\\
&\nu_2(z) = \sigma(z), && \nu_2(y) = \lambda^{-1}y,\\
&\nu_3(z) = z, && \nu_3(x) = \lambda x, && \nu_3(y) = \lambda^{-1}y,
\end{aligned}
$$

respectively. Thus the lemma follows from Rees Lemma. $\square$

Theorem 4.3. *Let A, B, $\varphi(z)$ be as above. The following are equivalent:*

(1) $\operatorname{Hcdim} A = 2$*;*
(2) *the global dimension of A is finite;*
(3) *$\varphi(z)$ has no multiple roots.*

When these conditions are satisfied, A is ν-twisted Calabi-Yau of dimension 2.

Proof. It follows from Lemmas 4.1 and 4.2 that $\operatorname{Hcdim} A = 2$ and A is ν-twisted Calabi-Yau if $\varphi(z)$ has no multiple roots.

Conversely, the global dimension of A is infinity if $\varphi(z) = 0$ or has multiple roots, proved in [2]. □

Quantum 2-spheres are a continuously parametrized family of $SU_q(2)$-spaces that are analogs of the classical 2-sphere $SU(2)/SO(2)$. They were firstly constructed by Podleś [14], and later studied by many other people, see [3], [7], [8], [10], [13], etc. Quantum 2-spheres are a class of important quantum homogeneous spaces whose many properties are discovered and applied in the realms of quantum group, noncommutative geometry, and mathematical physics.

As a $\mathbf{k}$-algebra, quantum 2-sphere is generated by X, Y, Z, subject to

$$\begin{aligned} &XZ = q^2 ZX, && ZY = q^2 YZ, \\ &YX = uv + (u-v)Z - Z^2, && XY = uv + q^2(u-v)Z - q^4 Z^2, \end{aligned}$$

where $\mathbf{k}$ is of characteristic zero, $0 \neq q \in \mathbf{k}$ is not a root of unity, and u, $v \in \mathbf{k}$ with $u + v \neq 0$. Usually, $\mathbf{k} = \mathbb{C}$ and q, u, $v \in \mathbb{R}$. It is a generalized Weyl algebra with $B = \mathbf{k}[Z]$, $\sigma(Z) = q^2 Z$, and $\varphi(Z) = uv + (u-v)Z - Z^2$. Note that $\varphi(Z)$ has two distinct roots u, $-v$. A quantum sphere is called standard if $(u, v) = (1, 0)$, which turns out to be homologically smooth [10]. The following corollary is obviously from Theorem 4.3 and hence replies in the affirmative to [10, Question 2].

Corollary 4.4. *The quantum 2-spheres are all homologically smooth.*

5. Noncommutative deformations

We study the noncommutative deformations of GWA in the section. Suppose furthermore that the characteristic of $\mathbf{k}$ is zero and σ is not the identity map, except in subsection 5.2.

We first put an interpretation how to view A as a deformation of some commutative algebra. After that, the deformations of the homologically smooth A will be computed.

5.1. **GWA—deformation of a commutative algebra.** We need to treat the field $\mathbf{k}$ and pose assumptions on the parameters λ, η, a_i. The idea is motivated by [16] (also see [12]).

Denote by $\mathbf{k}_0$ the subfield of $\mathbf{k}$ generated by $a_0, \ldots, a_l$ over $\mathbb{Q}$. Let t be equal to $\lambda - 1$ in the quantum case, or η in the classical case. We put forward two assumptions:
Assu 1: t is a transcendental element over $\mathbf{k}_0$,
Assu 2: the formal Laurent series field $\mathbf{k}_0(\!(t)\!)$ is a subfield of $\mathbf{k}$.

Under these assumptions, consider the $\mathbf{k}_0[[t]]$-algebra $\tilde{A}$ whose generators are x, y, z, subject to $yx = \varphi(z)$ as well as

$$\begin{cases} [x,z] = tzx, \\ [z,y] = tyz, \\ [x,y] = \varphi(z+tz) - \varphi(z) = tz\varphi'(z) + O(t^2), \end{cases}$$

or

$$\begin{cases} [x,z] = tx, \\ [z,y] = ty, \\ [x,y] = \varphi(z+t) - \varphi(z) = t\varphi'(z) + O(t^2), \end{cases}$$

depending on the quantum or classical case.

The algebra $\tilde{A}$ is complete with respect to the t-adic topology, $\mathbf{k}_0[[t]]$-flat, and so $\tilde{A}$ is a deformation of the commutative algebra

$$\mathfrak{A} := \tilde{A}/t\tilde{A} = \mathbf{k}_0[x,y,z]/(yx - \varphi(z)).$$

The deformation determines the Poisson brackets

$$\begin{aligned} &\text{quantum case:} \quad \{x,z\} = zx,\ \{z,y\} = yz,\ \{x,y\} = z\varphi'(z), \\ &\text{classical case:} \quad \{x,z\} = x,\ \ \{z,y\} = y,\ \ \{x,y\} = \varphi'(z). \end{aligned}$$

It is easy to check $A = \mathbf{k} \otimes_{\mathbf{k}_0[[t]]} \tilde{A}$. For simplicity, we may even assume $\mathbf{k} = \mathbf{k}_0(\!(t)\!)$. In this case A is the localization of $\tilde{A}$ at t.

5.2. **Van den Bergh duality.** Deformation of an associative algebra is closely related to the Hochschild cohomology $H^2(A,A)$. The Hochschild cohomology of GWA (without the homological smoothness condition) has been computed in [4] (classical case) and [15] (quantum case), especially in [15] the authors obtain a nice dimension formula when $z \nmid \varphi(z)$. However, for the purpose of studying deformation, their results are not satisfactory. [4] only tells us the dimensions of each $H^\bullet(A,A)$, without the cocycles; the condition $z \nmid \varphi(z)$ seems a little bit unnatural in [15] because it excludes some algebras we are familiar with; and both papers require $\deg \varphi(z) \geq 1$. In fact, the crucial role played by $z \nmid \varphi(z)$ is to suppress the appearance of (z) in the decomposition of $(\varphi(z))$. By the explanation in subsection 2.4, (z) is a fixed point in the space $\operatorname{MSpec} \mathbf{k}[z]$. It might have bad behavior during the computation.

We will remedy the defect here. Of course, it is unwise to compute all of the exceptional cases; that would be a burdensome job. Instead, we would rather first establish the Van den Bergh duality, and then compute $H^2(A, A)$ by a unified method. In this way, we may deal with the situations $z \mid \varphi(z)$ and $\deg \varphi(z) = 0$ more easily.

Suppose that $(m_1, m_2, m_3, m_4) \in \operatorname{Ker} \partial^2$, then

$$\partial_h^{11}(m_1, m_2) + \partial_v^2(m_3, m_4) = 0,$$
$$s^1(m_1, m_2) + \partial_h^{20}(m_3, m_4) = 0.$$

It follows that

$$-ym_1 - \lambda m_2 x + zm_3 - m_3 z = 0, \tag{5.1}$$
$$-m_1 y - \lambda x m_2 + \sigma(z)m_4 - m_4\sigma(z) = 0, \tag{5.2}$$
$$-\Delta(\varphi) \cdot m_1 + xm_3 - m_4 x = 0, \tag{5.3}$$
$$-\lambda\Delta(\varphi) \cdot m_2 - m_3 y + ym_4 = 0. \tag{5.4}$$

By induction, we obtain from (5.1), (5.2) that for any $j \geq 1$,

$$z^j m_3 - m_3 z^j = y(\Delta(z^j) \cdot m_1) + \lambda(\Delta(z^j) \cdot m_2)x,$$
$$\sigma(z)^j m_4 - m_4\sigma(z)^j = (\Delta(z^j) \cdot m_1)y + \lambda x(\Delta(z^j) \cdot m_2).$$

Thus

$$\Delta(\varphi) \cdot (m_3\beta) = m_3\beta\varphi' + y(\Delta^D(\varphi) \cdot m_1)\beta + \lambda(\Delta^D(\varphi) \cdot m_2)\sigma(\beta)x, \tag{5.5}$$
$$\Delta(\varphi) \cdot (m_4\sigma(\beta)) = m_4\sigma(\beta\varphi') + (\Delta^D(\varphi) \cdot m_1)\beta y + \lambda x(\Delta^D(\varphi) \cdot m_2)\sigma(\beta). \tag{5.6}$$

Let $n_1 = -m_3\beta \in M$. Then by (5.5) the first component of $s^0(n_1)$ is

$$\begin{aligned} &m_3\tilde{\varphi} - m_3\alpha yx + y(\Delta^D(\varphi) \cdot m_1)\beta + \lambda(\Delta^D(\varphi) \cdot m_2)\sigma(\beta)x \\ &= m_3\tilde{\varphi} + y(\Delta^D(\varphi) \cdot m_1)\beta - \big(m_3\alpha y - \lambda(\Delta^D(\varphi) \cdot m_2)\sigma(\beta)\big)x. \end{aligned}$$

Denote

$$n_3 = -(\Delta^D(\varphi) \cdot m_1)\beta \in {}^{\sigma}M,$$
$$n_4 = m_3\alpha y - \lambda(\Delta^D(\varphi) \cdot m_2)\sigma(\beta) \in M^{\sigma}.$$

So the first component of $s^0(n_1) + \partial_h^{10}(n_3, n_4)$ is $m_3\tilde{\varphi}$.

Next we will compute $\partial_h^{01}(n_1) + \partial_v^1(n_3, n_4)$. Recall (4.7), and thus

$$\sigma(z)n_3 - n_3 z = m_1\beta\varphi' - (\Delta(\varphi) \cdot m_1)\beta,$$
$$zn_4 - n_4\sigma(z) = zm_3\alpha y - m_3\alpha y\sigma(z) + \lambda m_2\sigma(\beta\varphi') - \lambda(\Delta(\varphi) \cdot m_2)\sigma(\beta).$$

So by (5.1)–(5.4),

$$\begin{aligned} &\partial_h^{01}(n_1) + \partial_v^{10}(n_3, n_4) \\ &= \big(xm_3\beta - \lambda m_3\beta x + m_1\beta\varphi' - (\Delta(\varphi) \cdot m_1)\beta,\ ym_3\beta - \lambda^{-1}m_3\beta y \\ &\quad + \lambda^{-1}zm_3\alpha y - \lambda^{-1}m_3 z\alpha y + m_2\sigma(\beta\varphi') - (\Delta(\varphi) \cdot m_2)\sigma(\beta)\big) \end{aligned}$$

$$
\begin{aligned}
&= \big(m_1\beta\varphi' + m_4x\beta - \lambda m_3\beta x,\ ym_3\beta - \lambda^{-1}ym_4\sigma(\beta) + \lambda^{-1}zm_3\alpha y \\
&\quad - \lambda^{-1}m_3z\alpha y + m_2\sigma(\tilde{\varphi}) - m_2x\alpha y\big) \\
&= \big(m_1\beta\varphi' + m_4x\beta - \lambda m_3\beta x,\ ym_3\beta - \lambda^{-1}ym_4\sigma(\beta) + \lambda^{-1}ym_1\alpha y \\
&\quad + m_2\sigma(\tilde{\varphi})\big) \\
&= \big(m_1\tilde{\varphi},\ m_2\sigma(\tilde{\varphi})\big) - \big(m_1\alpha yx + \lambda m_3\beta x - m_4\sigma(\beta)x,\ -\lambda^{-1}ym_1\alpha y \\
&\quad - ym_3\beta + \lambda^{-1}ym_4\sigma(\beta)\big).
\end{aligned}
$$

Denote $n_2 = \lambda^{-1}m_1\alpha y + m_3\beta - \lambda^{-1}m_4\sigma(\beta) \in M$. Then

$$(m_1, m_2)\cdot\tilde{\varphi} = \partial_h^{01}(n_1) + \partial_v^{1}(n_3, n_4) + (\lambda n_2x, -yn_2).$$

Finally, we need to compute $(m_3, m_4)\cdot\tilde{\varphi} - s^0(n_1) - \partial_h^{10}(n_3, n_4)$. To the end, it suffices to consider the second component. By (5.3), (5.6),

$$
\begin{aligned}
&m_4\sigma(\tilde{\varphi}) + \lambda^{\sigma}\Delta^{\sigma}(\varphi)\cdot n_1 - n_3y - xn_4 \\
&= m_4\sigma(\tilde{\varphi}) - \lambda^{\sigma}\Delta^{\sigma}(\varphi)\cdot(m_3\beta) + (\Delta^{D}(\varphi)\cdot m_1)\beta y - xm_3\alpha y \\
&\quad + \lambda x(\Delta^{D}(\varphi)\cdot m_2)\sigma(\beta) \\
&= m_4\sigma(\tilde{\varphi}) - {}^{\sigma}\Delta^{\sigma}(\varphi)\cdot(\lambda n_2 - m_1\alpha y + m_4\sigma(\beta)) + (\Delta^{D}(\varphi)\cdot m_1)\beta y \\
&\quad - xm_3\alpha y + \lambda x(\Delta^{D}(\varphi)\cdot m_2)\sigma(\beta) \\
&= m_4\sigma(\tilde{\varphi}) - \lambda^{\sigma}\Delta^{\sigma}(\varphi)\cdot n_2 + (\Delta(\varphi)\cdot m_1)\alpha y - m_4\sigma(\beta\varphi') - xm_3\alpha y \\
&= m_4\sigma(\alpha)xy - \lambda^{\sigma}\Delta^{\sigma}(\varphi)\cdot n_2 - m_4x\alpha y \\
&= -\lambda^{\sigma}\Delta^{\sigma}(\varphi)\cdot n_2,
\end{aligned}
$$

so $(m_3, m_4)\cdot\tilde{\varphi} = s^0(n_1) + \partial_h^{10}(n_3, n_4) + (0, -\lambda^{\sigma}\Delta^{\sigma}(\varphi)\cdot n_2)$. It follows that

$$(m_1, m_2, m_3, m_4)\cdot\tilde{\varphi} = \partial^1(n_1, n_3, n_4) + (\lambda n_2x, -yn_2, 0, -\lambda^{\sigma}\Delta^{\sigma}(\varphi)\cdot n_2).$$

Define two maps $f\colon M \to {}^{\sigma}M \oplus M^{\sigma} \oplus M \oplus {}^{\sigma}M^{\sigma}$, $g\colon \operatorname{Ker}\partial^2 \to M$ by

$$
\begin{aligned}
f(m) &= (\lambda mx, -ym, 0, -\lambda^{\sigma}\Delta^{\sigma}(\varphi)\cdot m), \\
g(m_1, m_2, m_3, m_4) &= \lambda^{-1}m_1\alpha y + m_3\beta - \lambda^{-1}m_4\sigma(\beta).
\end{aligned}
$$

One can directly verify $f(m)$ is a 2-cocycle. Hence f is a map from M to $\operatorname{Ker}\partial^2$. Denote by $\tilde{\varphi}_r$ and $\tilde{\Phi}_r$ the right multiplications determined by $\tilde{\varphi}(z)$ on M and $\operatorname{Ker}\partial^2$ respectively.

Lemma 5.1. *One has*

(1) $f([A, M^{\nu}]) \subseteq \operatorname{Im}\partial^1$,
(2) $g(\operatorname{Im}\partial^1) \subseteq [A, M^{\nu}]$,
(3) $fg - \tilde{\Phi}_r \subseteq \operatorname{Im}\partial^1$,
(4) $gf - \tilde{\varphi}_r \subseteq [A, M^{\nu}]$.

Proof. Part (3) follows from the definitions of f and g directly. Parts (1) and (4) are easy to check. For Part (2), the first, third, fourth components of $\partial^1(m_1, m_2, m_3)$ are $-xm_1+\lambda m_1 x+\sigma(z)m_2-m_2 z$, $-\Delta(\varphi)\cdot m_1+ym_2+m_3x$, $-\lambda^\sigma\Delta^\sigma(\varphi)\cdot m_1+m_2y+xm_3$, respectively. Thus

$$\begin{aligned}
&g(\partial^1(m_1, m_2, m_3))\\
={}&\lambda^{-1}(-xm_1+\lambda m_1 x+\sigma(z)m_2-m_2z)\alpha y+(-\Delta(\varphi)\cdot m_1+ym_2\\
&+m_3x)\beta-\lambda^{-1}(-\lambda^\sigma\Delta^\sigma(\varphi)\cdot m_1+m_2y+xm_3)\sigma(\beta)\\
={}&-\lambda^{-1}xm_1\alpha(z)y+m_1x\alpha y+zm_2\alpha y-m_2\alpha yz-(\Delta(\varphi)\cdot m_1)\beta\\
&+ym_2\beta+m_3x\beta+({}^\sigma\Delta^\sigma(\varphi)\cdot m_1)\sigma(\beta)-\lambda^{-1}m_2\beta y-\lambda^{-1}xm_3\sigma(\beta)\\
={}&[x,-\lambda^{-1}m_1\alpha y]-m_1\alpha yx+m_1x\alpha y+[z,m_2\alpha y]-(\Delta(\varphi)\cdot m_1)\beta\\
&+[y,m_2\beta]+({}^\sigma\Delta^\sigma(\varphi)\cdot m_1)\sigma(\beta)-[x,\lambda^{-1}m_3\sigma(\beta)]\\
={}&[x,-\lambda^{-1}m_1\alpha y]+[z,m_2\alpha y]-[z,(\Delta^D(\varphi)\cdot m_1)\beta]+[y,m_2\beta]\\
&+[z,\lambda({}^\sigma\Delta^{\sigma D}(\varphi)\cdot m_1)\sigma(\beta)]-[x,\lambda^{-1}m_3\sigma(\beta)]\\
\in{}&[A,M^\nu].
\end{aligned}$$

□

Proposition 5.2. *If A is homologically smooth, i.e., $\tilde{\varphi}(z)=1$, then $f\colon M\to \operatorname{Ker}\partial^2$ induces an isomorphism from $H_0(A,M^\nu)$ to $H^2(A,M)$, with inverse induced by g.*

5.3. **Computing $H_0(A,A^\nu)$.** It is known that $H_0(A,A^\nu)=A/[A,A^\nu]$. For any $K\in A$, denote by $[\![K]\!]$ the homology class presented by K in $A/[A,A^\nu]$.

5.3.1. *Classical case.* Only need to consider the case $\sigma(z)=z+1$.

For any $i\geq 0$, $j>0$, $[\![z^{i+1}x^j]\!]=[\![z^ix^jz]\!]=[\![z^i(z+j)x^j]\!]=[\![z^{i+1}x^j]\!]+j[\![z^ix^j]\!]$. So $[\![z^ix^j]\!]=0$. And similarly $[\![z^iy^j]\!]=0$. Since $[\![yxz^n]\!]=[\![xz^ny]\!]=[\![xy(z+1)^n]\!]$, we have $\sum_{i=0}^l a_i[\![z^{i+n}]\!]=\sum_{i=0}^l a_i[\![(z+1)^{i+n}]\!]$, and then develop it, obtaining

$$a_l(n+l)[\![z^{n+l-1}]\!]=\sum_{j=0}^{n+l-2}c_{nj}[\![z^j]\!]$$

with some coefficients $c_{nj}\in\mathbf{k}$. So for any n, $[\![z^{n+l-1}]\!]$ is a linear combination of $[\![1]\!],[\![z]\!],\ldots,[\![z^{n+l-2}]\!]$. Hence $[\![z^n]\!]$ can be linearly expressed by $[\![1]\!],[\![z]\!],\ldots,[\![z^{l-2}]\!]$.

Therefore,

$$A/[A,A^\nu]=\begin{cases}0, & \text{if } l=0,\ 1,\\ \displaystyle\bigoplus_{i=0}^{l-2}\mathbf{k}[\![z^i]\!], & \text{if } l\geq 2.\end{cases}\tag{5.7}$$

5.3.2. *Quantum case.* Suppose $\sigma(z) = \lambda z$ with $\lambda \neq 0$, 1. Denote $R = R(\varphi, \lambda)$ and recall that e is the order of λ.

First of all, for any i, $j > 0$, $[\![x^j]\!] = [\![x^{j-1}\nu(x)]\!] = \lambda[\![x^j]\!]$. Also, $[\![z^i x^j]\!] = [\![z^{i-1}x^j z]\!] = \lambda^j[\![z^i x^j]\!]$, and $[\![z^i x^j]\!] = \lambda^{-1}[\![xz^i x^{j-1}]\!] = \lambda^{i-1}[\![z^i x^j]\!]$. So $[\![z^i x^j]\!] \neq 0$ if $i \in e\mathbb{N}+1$, $j \in e\mathbb{N}$. An analogous discussion holds for $[\![z^i y^j]\!]$.

It suffices to consider $[\![z^n]\!]$ for $n \in \mathbb{N}$. It follows from $[\![yxz^n]\!] = \lambda^{-1}[\![xz^n y]\!] = \lambda^{n-1}[\![xyz^n]\!]$ that

$$\sum_{i=0}^{l} a_i(1-\lambda^{i+n-1})[\![z^{i+n}]\!] = 0.$$

Let $S_n = (1-\lambda^{n-1})[\![z^n]\!]$. Then $\{S_n\}_{n\in\mathbb{N}}$ is a linear recursive sequence. Suppose the (distinct) roots of $\varphi(z)$ are $z_1, \ldots, z_l$. Then we have

$$(S_0, S_1, \ldots) = (T_0, \ldots, T_{l-1})V(z_1, \ldots, z_l) \tag{5.8}$$

where $T_i \in A/[A, A^\nu]$.

Since $S_{ke+1} = 0$ for all $k \geq 0$, it follows from (5.8) that

$$(T_0, T_1, \ldots, T_{l-1})V_\lambda(z_1, \ldots, z_l) = 0.$$

Observe that R equals the number of distinct nonzero values in $z_1^e, \ldots, z_l^e$. So, by rearranging the order of the roots if necessary, we may assume that $z_1 = 0$ if $z \mid \varphi(z)$, and $z_{l-R+1}^e, \ldots, z_{l-1}^e, z_l^e$ are distinct. In this way, $\{T_0, \ldots, T_{l-R-1}\}$ is a maximal linearly independent subset of $\{T_0, \ldots, T_{l-1}\}$.

Define $\xi\colon \mathbb{N}^+ \to \mathbb{N}$ by $\xi(1) = 0$ and

$$\begin{aligned} &\xi(k(e-1)+c) = ke + c \text{ for all } k \geq 0 \text{ and } 2 \leq c \leq e, && \text{if } e \geq 2,\\ &\xi(n) = n \text{ for all } n \geq 2, && \text{if } e = 0. \end{aligned}$$

Then by (5.8), $\{S_{\xi(1)}, \ldots, S_{\xi(l-R)}\}$ is a maximal linearly independent subset of $\{S_0, S_1, \ldots\}$. Consequently,

$$A/[A, A^\nu] = \bigoplus_{i=1}^{l-R} \mathbf{k}[\![z^{\xi(i)}]\!] \oplus \bigoplus_{j\in e\mathbb{N}} \bigoplus_{k\in e\mathbb{Z}} \mathbf{k}[\![z^{j+1}x_k]\!]. \tag{5.9}$$

In particular, when $e = 0$, this is a finite dimensional vector space. Specifically, we have

$$A/[A, A^\nu] = \begin{cases} \mathbf{k}[\![z]\!], & \text{if } l = 0, \text{ or } (\, l = 1 \text{ and } a_0 \neq 0\,),\\ \mathbf{k}[\![1]\!] \oplus \mathbf{k}[\![z]\!], & \text{if } l = 1 \text{ and } a_0 = 0,\\ \displaystyle\bigoplus_{i=0}^{l-1} \mathbf{k}[\![z^i]\!], & \text{if } l \geq 2. \end{cases}$$

Finally, we should mention that the above deduction is under the condition that all roots of $\varphi(z)$ lie in $\mathbf{k}$. However, this is just for finding a maximal linearly independent

set, which does not depend on the field $\mathbf{k}$. So (5.9) is now valid for arbitrary $\mathbf{k}$, even when $\varphi(z)$ has no roots in $\mathbf{k}$.

5.4. **Hochschild cocycles.** Since $H^2(A, A)$ is known, we wish to find the corresponding Hochschild 2-cocycles. So let us construct two comparisons between the bar resolution $C^{\mathrm{bar}}_{\cdot}$ of A and $\operatorname{Tot}\mathcal{P}_{\cdot\cdot}$.

$$\begin{tikzcd}
A^{\otimes 2} \arrow[d, shift right, "\theta_0"'] & A^{\otimes 3} \arrow[l, "b'"'] \arrow[d, shift right, "\theta_1"'] & A^{\otimes 4} \arrow[l, "b'"'] \arrow[d, shift right, "\theta_2"'] & A^{\otimes 5} \arrow[l, "b'"'] \arrow[d, shift right, "\theta_3"'] & \cdots \arrow[l, "b'"'] \\
\mathcal{P}_{00} \arrow[u, shift right, "\theta'_0"'] & \mathcal{P}_{01} \oplus \mathcal{P}_{10} \arrow[l, "d_1"'] \arrow[u, shift right, "\theta'_1"'] & \mathcal{P}_{11} \oplus \mathcal{P}_{20} \arrow[l, "d_2"'] \arrow[u, shift right, "\theta'_2"'] & \mathcal{P}_{21} \oplus \mathcal{P}_{30} \arrow[l, "d_3"'] \arrow[u, shift right, "\theta'_3"'] & \cdots \arrow[l] .
\end{tikzcd}$$

In order to save space, we write $a_1|a_2|\dots|a_n$ instead of $a_1 \otimes a_2 \otimes \cdots \otimes a_n$. Clearly, $\theta_0 = \theta'_0 = \mathrm{id}$. And we define θ_1, θ'_1 by

$$\theta_1(1|z^i x^j|1) = \left(\sum_{k=1}^{i} z^{i-k}|z^{k-1}x^j,\ \sum_{k=1}^{j} z^i x^{j-k}|x^{k-1},\ 0 \right),$$
$$\theta_1(1|z^i y^j|1) = \left(\sum_{k=1}^{i} z^{i-k}|z^{k-1}y^j,\ 0,\ \sum_{k=1}^{j} z^i y^{j-k}|y^{k-1} \right),$$

and

$$\theta'_1(1|1,0,0) = 1|z|1, \quad \theta'_1(0,1|1,0) = 1|x|1, \quad \theta'_1(0,0,1|1) = 1|y|1.$$

Recall that

$$\begin{aligned}
d_2(1|1,0,0,0) &= (-x|1 + \lambda|x,\ \sigma(z)|1 - 1|z,\ 0),\\
d_2(0,1|1,0,0) &= (-y|1 + \lambda^{-1}|y,\ 0,\ \lambda^{-1}z|1 - \lambda^{-1}|\sigma(z)),\\
d_2(0,0,1|1,0) &= (-\Delta(\varphi),\ y|1,\ 1|x),\\
d_2(0,0,0,1|1) &= (-\lambda^{\sigma}\Delta^{\sigma}(\varphi),\ 1|y,\ x|1).
\end{aligned}$$

By the fact $\Delta(\sigma^q(z)^i) = \lambda^q \sum_{k=1}^{i} \sigma^q(z)^{i-k}|\sigma^q(z)^{k-1}$, we have

$$\begin{aligned}
&\theta_1 b'(1|z^p x^q|z^i x^j|1)\\
&= \underbrace{\theta_1(z^p x^q|z^i x^j|1)}_{\text{Part 1}} - \underbrace{\theta_1(1|z^p\sigma^q(z)^i x^{q+j}|1)}_{\text{Part 2}} + \underbrace{\theta_1(1|z^p x^q|z^i x^j)}_{\text{Part 3}}
\end{aligned}$$

and the three parts are

$$\text{Part 1} = \left(\sum_{k=1}^{i} z^p x^q z^{i-k}|z^{k-1}x^j,\ \sum_{k=1}^{j} z^p x^q z^i x^{j-k}|x^{k-1},\ 0 \right),$$
$$\text{Part 2} = \left(\sum_{k=1}^{p} z^{p-k}|z^{k-1}\sigma^q(z)^i x^{q+j} + \sum_{k=1}^{i} \lambda^q z^p \sigma^q(z)^{i-k}|\sigma^q(z)^{k-1}x^{q+j}, \right.$$

$$\sum_{k=1}^{q+j} z^p\sigma(z)^i x^{q+j-k}|x^{k-1},\, 0\Big),$$

$$\text{Part 3} = \Big(\sum_{k=1}^{p} z^{p-k}|z^{k-1}x^q z^i x^j,\, \sum_{k=1}^{q} z^p x^{q-k}|x^{k-1}z^i x^j,\, 0\Big),$$

respectively. Thus a direct computation shows that the sum is equal to

$$d_2\Big(-\sum_{k=1}^{i}\sum_{s=1}^{q} z^p\sigma^q(z)^{i-k}x^{q-s}|(\lambda x)^{s-1}z^{k-1}x^j, 0, 0, 0\Big).$$

Denote $\Delta^\nu(x^q) = \sum_{s=1}^{q} x^{q-s}|(\lambda x)^{s-1}$ and define

$$\theta_2(1|z^p x^q|z^i x^j|1) = \big(-z^p({}^{\sigma^q}\Delta(z^i)\cdot\Delta^\nu(x^q))x^j,\, 0,\, 0,\, 0\big).$$

Similarly, we define

$$\begin{aligned}
\theta_2(1|z^p x|z^i y^j|1) &= \big(-z^p\, {}^{\sigma}\Delta(z^i)y^j,\, 0,\, 0,\, z^p\sigma(z^i)|y^{j-1}\big),\\
\theta_2(1|z^p y|z^i x^j|1) &= \big(0,\, -z^p\, {}^{\sigma^{-1}}\Delta(z^i)x^j,\, z^p\sigma^{-1}(z^i)|x^{j-1},\, 0\big),\\
\theta_2(1|z^p y^q|z^i y^j|1) &= \big(0,\, -z^p({}^{\sigma^{-q}}\Delta(z^i)\cdot\Delta^\nu(y^q))y^j,\, 0,\, 0\big),\\
\theta_2(1|z^p|z^i x^j|1) &= \theta_2(1|z^p|z^i y^j|1) = 0
\end{aligned}$$

where $\Delta^\nu(y^q) = \sum_{s=1}^{q} y^{q-s}|(\lambda^{-1}y)^{s-1}$.

It is hard to give explicit forms of $\theta(1|z^p x^q|z^i y^j|1)$ and $\theta(1|z^p y^q|z^i x^j|1)$ for $q \geq 2$. But we are only interested in cocycles here, so the above formulas are enough.

The morphisms θ_2', θ_3' are listed as follows.

$$\begin{aligned}
\theta_2'(1|1, 0, 0, 0) &= \lambda|z|x|1 - 1|x|z|1,\\
\theta_2'(0, 1|1, 0, 0) &= \lambda^{-1}|z|y|1 - 1|y|z|1,\\
\theta_2'(0, 0, 1|1, 0) &= 1|y|x|1 + 1|1|1|\varphi(z) - \sum_{i=1}^{l}\sum_{j=1}^{i} a_i|z^{i-j}|z|z^{j-1},\\
\theta_2'(0, 0, 0, 1|1) &= 1|x|y|1 + 1|1|1|\sigma(\varphi(z)) - \sum_{i=1}^{l}\sum_{j=1}^{i} a_i|\sigma(z)^{i-j}|\lambda z|\sigma(z)^{j-1},\\
\theta_3'(1|1, 0, 0, 0) &= 1|z|y|x|1 - \lambda|y|z|x|1 + 1|y|x|z|1 + 1|z|1|1|\varphi(z)\\
&\quad + 1|1|1|z|\varphi(z) - \sum_{i=1}^{l}\sum_{j=1}^{i} a_i|z|z^{i-j}|z|z^{j-1},\\
\theta_3'(0, 1|1, 0, 0) &= 1|z|x|y|1 - \lambda^{-1}|x|z|y|1 + 1|x|y|z|1 + 1|z|1|1|\sigma(\varphi(z))\\
&\quad + 1|1|1|z|\sigma(\varphi(z)) - \sum_{i=1}^{l}\sum_{j=1}^{i} a_i|z|\sigma(z)^{i-j}|\lambda z|\sigma(z)^{j-1},
\end{aligned}$$

$$\theta_3'(0,0,1|1,0) = 1|x|y|x|1 - \sum_{i=1}^{l}\sum_{j=1}^{i}\Big(a_i|x|z^{i-j}|z|z^{j-1} - a_i|\sigma(z)^{i-j}|x|z|z^{j-1} + a_i|\sigma(z)^{i-j}|\lambda z|x|z^{j-1}\Big) + 1|x|1|1|\varphi(z) + 1|1|1|x|\varphi(z),$$

$$\theta_3'(0,0,0,1|1) = 1|y|x|y|1 - \sum_{i=1}^{l}\sum_{j=1}^{i}\Big(a_i|y|\sigma(z)^{i-j}|\lambda z|\sigma(z)^{j-1} - a_i|z^{i-j}|y|\lambda z|\sigma(z)^{j-1} + a_i|z^{i-j}|z|y|\sigma(z)^{j-1}\Big).$$

By the construction, $\theta_i\theta_i' = \mathrm{id}$ for $i = 0, 1, 2$. The first three terms of $\operatorname{Tot}\mathcal{P}_{..}$ are thus the direct summand of those of $C_{.}^{\mathrm{bar}}$, and the differentials d_i can be viewed as the restriction of b'.

Suppose that $F_K\colon A\times A\to A$ is the 2-cocycle corresponding to $f(K)\in\operatorname{Ker}\partial^2$ where K is any element in the vector space (5.7) or (5.9). Note that K commutes with z and $D = \mathrm{d}/\mathrm{d}z$. So

$$\begin{aligned}
F_K(z^px^q, z^ix^j) &= -\lambda z^p(\Delta^\nu(x^q)\cdot K)xD(z^i)x^j,\\
F_K(z^px, z^iy^j) &= -\lambda z^pKxD(z^i)y^j - z^pKD(\varphi\circ\sigma)\sigma(z)^iy^{j-1},\\
F_K(z^py, z^ix^j) &= z^pyKD(z^i)x^j,\\
F_K(z^py^q, z^iy^j) &= z^py(\Delta^\nu(y^q)\cdot K)D(z^i)y^j.
\end{aligned}$$

In order to give the explicit forms of $F_K(z^px^q, z^iy^j)$ and $F_K(z^py^q, z^ix^j)$ for $q\geq 2$, we use the condition $F_K(a_1a_2, a_3) = F_K(a_1, a_2a_3) + a_1F_K(a_2, a_3) - F_K(a_1, a_2)a_3$ for all $a_i\in A$. Notice

$$\begin{aligned}
x^kh(z)y^j &= xy\sigma^k(h(z))\sigma^k(\varphi(z))\cdots\sigma^2(\varphi(z))y^{j-k},\\
y^kh(x)x^j &= yx\sigma^{-k}(h(z))\sigma^{-k+1}(\varphi(z))\cdots\sigma^{-1}(\varphi(z))x^{j-k}
\end{aligned}$$

for $1\leq k\leq j$ and $h(z)\in\mathbf{k}[z]$. We introduce the formal notations as follows,

$$\begin{aligned}
(xy)^{-1}(x^kh(z)y^j) &:= \sigma^k(h(z))\sigma^k(\varphi(z))\cdots\sigma^2(\varphi(z))y^{j-k},\\
(yx)^{-1}(y^kh(x)x^j) &:= \sigma^{-k}(h(z))\sigma^{-k+1}(\varphi(z))\cdots\sigma^{-1}(\varphi(z))x^{j-k}.
\end{aligned}$$

Lemma 5.3. *Let $\gamma = \min\{q-1, j\}$. One has*

$$\begin{aligned}
F_K(z^px^q, z^iy^j) &= -\lambda z^p(\Delta^\nu(x^q)\cdot K)xD(z^i)y^j\\
&\quad - \sum_{k=1}^{\gamma+1} z^p(\Delta^\nu(x^{q-k+1})\cdot K)D(\varphi\circ\sigma)(xy)^{-1}(x^kz^iy^j),\\
F_K(z^py^q, z^ix^j) &= z^py(\Delta^\nu(y^q)\cdot K)D(z^i)x^j\\
&\quad + \sum_{k=1}^{\gamma} z^py(\Delta^\nu(y^{q-k})\cdot K)D(\varphi)(yx)^{-1}(y^kz^ix^j).
\end{aligned}$$

Proof. Both formulas can be proved by induction on q. We only prove the first here.

This is true for $q = 0$, 1 and for $j = 0$. Now suppose $q \geq 2$, $j \geq 1$, and by hypothesis,

$$F_K(z^p x^{q-1}, z^i y^{j-1}) = -\lambda z^p(\Delta^\nu(x^{q-1}) \cdot K)xD(z^i)y^{j-1} - \sum_{k=1}^{\gamma} z^p(\Delta^\nu(x^{q-k}) \cdot K)D(\varphi \circ \sigma)(xy)^{-1}(x^k z^i y^{j-1}).$$

Since $F_K(z^p x^{q-1}, x) = 0$, it follows that

$$\begin{aligned}
&F_K(z^p x^q, z^i y^j) = F_K(z^p x^{q-1}, xz^i y^j) + z^p x^{q-1} F_K(x, z^i y^j)\\
&= F_K(z^p x^{q-1}, \sigma(z)^i(\varphi \circ \sigma)(z)y^{j-1}) - \lambda z^p x^{q-1} KxD(z^i)y^j\\
&\quad - z^p x^{q-1} KD(\varphi \circ \sigma)\sigma(z)^i y^{j-1}\\
&= -\lambda z^p(\Delta^\nu(x^{q-1}) \cdot K)xD(\sigma(z)^i(\varphi \circ \sigma)(z))y^{j-1}\\
&\quad - \sum_{k=1}^{\gamma} z^p(\Delta^\nu(x^{q-k}) \cdot K)D(\varphi \circ \sigma)(xy)^{-1}(x^k \sigma(z)^i(\varphi \circ \sigma)(z)y^{j-1})\\
&\quad - \lambda z^p x^{q-1} KxD(z^i)y^j - z^p x^{q-1} KD(\varphi \circ \sigma)\sigma(z)^i y^{j-1}\\
&= -\lambda z^p(\Delta^\nu(x^{q-1}) \cdot K)x(\lambda xD(z^i)y + \sigma(z)^i D(\varphi \circ \sigma))y^{j-1}\\
&\quad - \sum_{k=1}^{\gamma} z^p(\Delta^\nu(x^{q-k}) \cdot K)D(\varphi \circ \sigma)(xy)^{-1}(x^{k+1} z^i y^j)\\
&\quad - \lambda z^p x^{q-1} KxD(z^i)y^j - z^p x^{q-1} KD(\varphi \circ \sigma)(xy)^{-1}(xz^i y^j)\\
&= -\lambda z^p(\Delta^\nu(x^q) \cdot K)xD(z^i)y^j - z^p(\Delta^\nu(x^q) \cdot K)D(\varphi \circ \sigma)(xy)^{-1}(xz^i y^j)\\
&\quad - \sum_{k=2}^{\gamma+1} z^p(\Delta^\nu(x^{q-k+1}) \cdot K)D(\varphi \circ \sigma)(xy)^{-1}(x^k z^i y^j)\\
&= -\lambda z^p(\Delta^\nu(y^q) \cdot K)xD(z^i)y^j\\
&\quad - \sum_{k=1}^{\gamma+1} z^p(\Delta^\nu(x^{q-k+1}) \cdot K)D(\varphi \circ \sigma)(xy)^{-1}(x^k z^i y^j).
\end{aligned}$$

□

The above results can be summarized as

Theorem 5.4. *Let A be a homologically smooth GWA, and let K be any element in the vector space* (5.7) *or* (5.9)*. The equivalence class of first order deformations of A corresponds to the 2-cocycle $F_K \colon A \times A \to A$ given by*

$$F_K(z^p x^q, z^i x^j) = -\lambda z^p(\Delta^\nu(x^q) \cdot K)xD(z^i)x^j,$$
$$F_K(z^p x^q, z^i y^j) = -\lambda z^p(\Delta^\nu(x^q) \cdot K)xD(z^i)y^j$$

$$\begin{aligned}
&\quad -\sum_{k=1}^{\gamma+1} z^p(\Delta^\nu(x^{q-k+1})\cdot K)D(\varphi\circ\sigma)(xy)^{-1}(x^k z^i y^j),\\
F_K(z^py^q, z^ix^j) &= z^py(\Delta^\nu(y^q)\cdot K)D(z^i)x^j\\
&\quad +\sum_{k=1}^{\gamma} z^py(\Delta^\nu(y^{q-k})\cdot K)D(\varphi)(yx)^{-1}(y^kz^ix^j),\\
F_K(z^py^q, z^iy^j) &= z^py(\Delta^\nu(y^q)\cdot K)D(z^i)y^j,
\end{aligned}$$

where $\gamma = \min\{q-1, j\}$.

Remark 5.5. Since we are interesting in "deformation of deformation", it seems that we should follow the assumptions in subsection 5.1. So it is unnecessary to consider the situation—λ is a root of unity. However, for the sake of completeness, these algebras are also considered.

The deformation when $K = z$ in the quantum case is very interesting. We will show that it offers a way "back to $\mathfrak{A}$".

Following subsection 5.1, we assume $\mathbf{k} = \mathbf{k}_0((t))$. Let $\sigma(z) = \lambda z$ and $F_1 = F_z$. Identify $F_1 \bullet F_1$ with a homomorphism in $\mathrm{Hom}_{A^e}(A^{\otimes 5}, A)$ and observe that the composition $(F_1 \bullet F_1)\circ\theta_3'$ corresponds to a cocycle in $Z^3(\mathrm{Tot}\,\mathcal{Q}^{\cdot\cdot})$ (recall the diagram in the beginning of Sect. 4, letting $M = A$). By the definition of θ_3', the cocycle is

$$\left(zyx,\ zxy,\ -\frac{1}{2}z^2D^2(\varphi\circ\sigma)x,\ yzD(\varphi\circ\sigma)+\frac{1}{2}yz^2D^2(\varphi\circ\sigma)\right).$$

We choose its preimage under ∂^2 to be

$$\left(0,\ -yz,\ 0,\ \frac{1}{2}z^2D^2(\varphi\circ\sigma)\right)$$

which corresponds to an A^e-module homomorphism $f_2\colon \mathcal{P}_{11}\oplus\mathcal{P}_{20} \to A$. Recall that we have proved $\mathcal{P}_{11}\oplus\mathcal{P}_{20}$ is a direct summand of $A^{\otimes 4}$. So we hope to find a $\mathbf{k}$-bilinear map $F_2\colon A\times A\to A$ extending f_2, that is, $f_2 = F_2\circ\theta_2'$. Hence we obtain a system of equations (notice $F_2(1,1) = 0$)

$$\begin{cases}
\lambda F_2(z,x) - F_2(x,z) &= 0\\
\lambda^{-1}F_2(z,y) - F_2(y,z) &= -yz\\
F_2(y,x) - \sum_{i=1}^{l}\sum_{j=1}^{i} a_iF_2(z^{i-j},z)z^{j-1} &= 0\\
F_2(x,y) - \sum_{i=1}^{l}\sum_{j=1}^{i} a_i\lambda^iF_2(z^{i-j},z)z^{j-1} &= \frac{1}{2}z^2D^2(\varphi\circ\sigma).
\end{cases} \tag{5.10}$$

Among the solutions, we choose one satisfying $F_2(z^i a_1, a_2) = z^i F_2(a_1, a_2)$. This is not amazing because F_1 has the same property. As a consequence,

$$F_2(z^i, -) = 0,\ F_2(x, z) = 0,\ F_2(y, z) = yz,$$
$$F_2(x, y) = \frac{1}{2} z^2 D^2(\varphi \circ \sigma),\ F_2(y, x) = 0.$$

Using $F_1 \bullet F_1 = \mathsf{d}F_2$, we moreover have

$$F_2(x, yz) = z^2 D(\varphi \circ \sigma) + \frac{1}{2} z^3 D^2(\varphi \circ \sigma),\ F_2(y, xz) = zyx,$$
$$F_2(x, h(z)) = \frac{1}{2} xz^2 D^2(h(z)),\ F_2(y, h(z)) = \frac{1}{2} yz D^2(zh(z)).$$

Next consider the map $F_1 \bullet F_2 + F_2 \bullet F_1$, whose composition with θ_3' corresponds to the cocycle

$$\Big(-zyx,\ -zxy,\ \frac{1}{6} z^3 D^3(\varphi \circ \sigma)x,\ -yzD(\varphi \circ \sigma) - \frac{1}{6} yz^3 D^3(\varphi \circ \sigma)\Big).$$

Choose a preimage to be

$$\Big(0,\ yz,\ 0,\ -\frac{1}{6} z^3 D^3(\varphi \circ \sigma)\Big),$$

and then establish a system of equations similar to (5.10), yielding that

$$F_3(z^i, -) = 0,\ F_3(x, z) = 0,\ F_3(y, z) = yz,$$
$$F_3(x, y) = -\frac{1}{6} z^3 D^3(\varphi \circ \sigma),\ F_3(y, x) = 0,$$
$$F_3(x, yz) = -\frac{1}{2} z^3 D^2(\varphi \circ \sigma) - \frac{1}{6} z^4 D^3(\varphi \circ \sigma),\ F_3(y, xz) = zyx,$$
$$F_3(x, h(z)) = -\frac{1}{6} xz^3 D^3(h(z)),\ F_3(y, h(z)) = \frac{1}{6} yz D^3(z^2 h(z)).$$

Continuing the procedure successively, and by the definitions of θ_2', θ_3', we may prove the following proposition by induction on n.

Proposition 5.6. *There exist a family of* $\mathbf{k}$*-bilinear maps* $F_n \colon A \times A \to A$, $n \geq 2$ *integrating* F_1 *that satisfy*

$$F_n(z^i, -) = 0,\ F_n(x, z) = 0,\ F_n(y, z) = yz,$$
$$F_n(x, y) = \frac{(-1)^n}{n!} z^n D^n(\varphi \circ \sigma),\ F_n(y, x) = 0,$$
$$F_n(x, yz) = \frac{(-1)^n}{(n-1)!} z^n D^{n-1}(\varphi \circ \sigma) + \frac{(-1)^n}{n!} z^{n+1} D^n(\varphi \circ \sigma),$$
$$F_n(y, xz) = zyx,$$
$$F_n(x, h(z)) = \frac{(-1)^n}{n!} xz^n D^n(h(z)),$$
$$F_n(y, h(z)) = \frac{1}{n!} yz D^n(z^{n-1} h(z)).$$

Henceforth, we obtain a formal deformation $(A[[\tau]], *)$ of A. Recall $\lambda = 1 + t$. Thus

$$\begin{aligned}
z * x &= zx,\ z * y = zy,\ z * z = z^2,\\
x * z &= xz - \tau xz = (1-\tau)xz = (1-\tau)(1+t)zx,\\
y * z &= \sum_{n=0}^{\infty} \tau^n yz = (1-\tau)^{-1} yz = (1-\tau)^{-1}(1+t)^{-1} zy,\\
x * y &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \tau^n z^n D^n(\varphi \circ \sigma)\\
&= (\varphi \circ \sigma)(z - \tau z) = \varphi\big((1-\tau)(1+t)z\big),\\
y * x &= yx = \varphi(z),
\end{aligned}$$

and so $A[[\tau]]$ is the quotient algebra of $\mathbf{k}_0(\!(t)\!)[[\tau]]\langle x, y, z\rangle$ by the relations

$$\begin{aligned}
x * z &= (1-\tau)(1+t) z * x,\\
z * y &= (1-\tau)(1+t) y * z,\\
x * y &= \varphi\big((1-\tau)(1+t)z\big),\\
y * x &= \varphi(z).
\end{aligned}$$

After evaluating $\tau = 1 - (1+t)^{-1} = -\sum_{n=1}^{\infty} (-t)^n$, $A[[\tau]]$ becomes a commutative $\mathbf{k}_0(\!(t)\!)$-algebra, isomorphic to $\mathbf{k}_0(\!(t)\!) \otimes_{\mathbf{k}_0} \mathfrak{A}$.

In the classical case, in order to find a return trip, the degree of $\varphi(z)$ must be at least two. Otherwise, the second Hochschild cohomology group of A is zero. An algebra enjoying such property is said to be absolutely rigid, whose any formal deformation is equivalent to the null one. The Weyl algebra $A_1(\mathbf{k})$ is a typical absolutely rigid algebra. So, although $A_1(\mathbf{k})$ is deformed by the commutative algebra $\mathbf{k}[x, y]$, it cannot return to $\mathbf{k}[x, y]$ via formal deformations.

The return trip exists indeed if $\deg \varphi(z) \geq 2$ provided that we start with $K = 1$. Thus

$$F_1(z, -) = 0,\ F_1(x, z) = -x,\ F_1(y, z) = y,$$
$$F_1(x, y) = -D(\varphi \circ \sigma),\ F_1(y, x) = 0.$$

By the same (but much easier) procedure, we have

Proposition 5.7. *There exist a family of* $\mathbf{k}$*-bilinear maps* $F_n \colon A \times A \to A$, $n \geq 2$ *integrating* F_1 *that satisfy*

$$F_n(z, -) = 0,\ F_n(x, z) = 0,\ F_n(y, z) = 0,$$
$$F_n(x, y) = \frac{(-1)^n}{n!} D^n(\varphi \circ \sigma),\ F_n(y, x) = 0.$$

Therefore, we have a formal deformation $(A[[\tau]], *)$ of A with

$$\begin{aligned}
&z * x = zx,\ z * y = zy,\ z * z = z^2,\\
&x * z = xz - \tau x = zx + (t-\tau)x,\\
&y * z = yz + \tau y = zy - (t-\tau)y,\\
&x * y = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\tau^n D^n(\varphi\circ\sigma)\\
&\qquad = (\varphi\circ\sigma)(z-\tau) = \varphi(z+t-\tau),\\
&y * x = yx = \varphi(z),
\end{aligned}$$

By evaluating $\tau = t$, then $A[[\tau]]$, as a commutative $\mathbf{k}_0(\!(t)\!)$-algebra, is isomorphic to $\mathbf{k}_0(\!(t)\!) \otimes_{\mathbf{k}_0} \mathfrak{A}$.

Acknowledgments

I am grateful to Prof. Wendy Lowen for her inspiring discussions and helpful conversations. Some ideas arose from my doctoral career. So I would like to thank my supervisor Prof. Quanshui Wu for his advice and encouragement.

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